Thermodynamics of Glaciers

Solutions to Exercise

1 Climate history

The initial amplitude is $\Delta T_0 = 2^\circ\,\mathrm{C}$ during an oscillation period of 50 years, thus $\omega = \frac{2\pi}{50\,\mathrm{a}} = 3.9821 \times 10^{-9}\,\mathrm{s}^{-1}$. With Equation (11) we get:

$$\frac{\Delta T(h)}{\Delta T_0} = \exp\left(-h\sqrt{\frac{\omega}{2\kappa}}\right).$$

The thermal diffusivity can be obtained from Equations (2) and (3)

$$\kappa(-3)^\circ\,\mathrm{C} = \frac{k(-3^\circ\,\mathrm{C})}{\rho\, c(-3^\circ\,\mathrm{C})} = \frac{2.1\,\mathrm{W\,m^{-1}\,K^{-1}}}{910\,\mathrm{kg\,m^{-3}} \cdot 2105.7\,\mathrm{J\,kg^{-1}\,K^{-1}}} = 1.34 \times 10^{-6}\,\mathrm{m^2\,s^{-1}},$$

or, just use the value given in the script, $1.09 \cdot 10^{-6}\,\mathrm{m^2\,s^{-1}}$ (the result then may slightly vary). Therefore we get

$$h = -\sqrt{\frac{2\kappa}{\omega}}\ln\left(\frac{\Delta T(h)}{\Delta T_0}\right) = -\sqrt{\frac{2 \cdot 1.34 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}}{3.9821 \times 10^{-9}\,\mathrm{s^{-1}}}}\ln\left(\frac{0.02}{2}\right) \approx 119\,\mathrm{m}.$$

2 Cold content

The cold content can be obtained by

$$E = -\int_0^z \rho c_p T(z)\,\mathrm{d}z'$$

This can be approximated by the area under the depth profile:

$$E \approx \frac{1}{2}\rho c_p \Delta T \Delta z = 105 \cdot 10^5\,\mathrm{J\,m^{-2}},$$

where $\rho = 500\,\mathrm{kg\,m^{-3}}$ and $c_p = 2009\,\mathrm{J\,kg^{-1}\,K^{-1}}$. Divide by the latent heat of fusion $L = 3.34 \cdot 10^5\,\mathrm{J\,kg^{-1}}$ and you get $32\,\mathrm{kg\,m^{-2}} = 32$ mm w.e..

3 Melting temperature depression

From Figure (9) we see that the pressure melting temperature is $T_\mathrm{m} = -0.34°\,\mathrm{C}$. We have $p = 910\,\mathrm{kg\,m^3} \cdot 9.81\,\mathrm{m\,s^{-2}} \cdot 300\,\mathrm{m} + 75'000\,\mathrm{Pa} = 2753130\,\mathrm{Pa}$. Using Equation (14) we obtain

$$\gamma = -\frac{T_\mathrm{m} - T_\mathrm{tp}}{p - p_\mathrm{tp}} = \frac{273.15\,\mathrm{K} - 0.34\,\mathrm{K} - 273.16\,\mathrm{K}}{2753130\,\mathrm{Pa} - 611.73\,\mathrm{Pa}} = 1.202 \times 10^{-7}\,\mathrm{K\,Pa^{-1}}.$$

In the literature we find the values $\gamma = 7.42 \times 10^{-8}\,\mathrm{K\,Pa^{-1}}$ for pure ice and air-free water and $\gamma = 9.8 \times 10^{-7}\,\mathrm{K\,Pa^{-1}}$ for pure ice and air-saturated water. The value calculated for Gornergletscher is even higher than γ for air-saturated water which means we have ice with air-saturated water.

4 Lake Vostok

1. Advection and diffusion
2. The Péclet number Pe is a measure of the relative importance of advection and diffusion.
3. First we calculate the pressure melting point at the base

$$\begin{aligned} T_m &= T_{tp} - \gamma\,(p - p_\mathrm{tp}) \\ &= 273.16\,\mathrm{K} - 7.42 \times 10^{-8}\,\mathrm{K\,Pa^{-1}}\left(910\,\mathrm{kg\,m^3} \cdot 9.81\,\mathrm{m\,s^{-2}} \cdot 3300\,\mathrm{m} - 611.73\,\mathrm{Pa}\right) \\ &\approx 271\,\mathrm{K} \end{aligned}$$

Then use Equation (17) from the script:

$$T(z) = T_s + \frac{\sqrt{\pi}}{2} l \left(\frac{dT}{dz}\right)_B \left[\mathrm{erf}\left(\frac{z}{l}\right) - \mathrm{erf}\left(\frac{H}{l}\right)\right].$$

In order for a lake to form we need $T(0\,\mathrm{m}) = T_\mathrm{m}$.

$$\begin{aligned} T(0\,\mathrm{m}) &= T_s + \frac{\sqrt{\pi}}{2} l \left(\frac{dT}{dz}\right)_B \left[\mathrm{erf}\left(\frac{z}{l}\right) - \mathrm{erf}\left(\frac{H}{l}\right)\right] \\ &= T_s + \frac{\sqrt{\pi}}{2} l \left(\frac{G}{k}\right) \left[\mathrm{erf}\left(\frac{z}{l}\right) - \mathrm{erf}\left(\frac{H}{l}\right)\right] \end{aligned}$$

The geothermal flux at the base thus must be equal or larger:

$$G \geq \frac{T_s - T(0\,\mathrm{m})}{\frac{\sqrt{\pi}}{2} l \left(\frac{1}{k}\right) \left[-\mathrm{erf}\left(\frac{H}{l}\right)\right]} \approx 0.05\,\mathrm{W\,m^2}$$