What are inverse problems?

A common situation in geophysics is the following: A seismic event occurs at a certain time and place. If we knew the characteristics of the seismic source (magnitude and type of event), as well as the velocity structure of the Earth, we could predict the response of a seismometer placed anywhere on the planet. Of course, we do not know any of these things; instead we can measure, to within a certain precision, the response of a seismometer. Figuring out the time, place, type, and magnitude of the source, as well as the velocity structure of the Earth forms perhaps the most classical inverse problem in Geophysics.

The following terms are commonly used in this context:

- **The forward model** $G$: The forward model is a mathematical representation of the relevant physics. In the above example it would be a theory of seismic wave propagation in the interior and along the surface of the planet. This theory always entails a certain amount of approximation and therefore cannot be expected to represent reality exactly.

- **The model parameters** $m$: The model parameters are the quantities we wish to know, but cannot determine directly. We can often constrain them, however. For example, seismic velocities have to be positive and within a certain range. The time of an earthquake cannot be later than when it was first measured, etc.

- **The data** $d$: The data are the quantities we can measure; in this example arrival times and waveforms at a discrete set of seismic stations. Data always come with errors; data stated without errors are meaningless.
The forward model $G$ could be used to determine $d$ if $m$ was known. That’s the forward problem, and it is well-posed: For given model parameters $m$ there are unique data $d$.

The inverse problem is to find model parameters $m$, given data $d$ and a model $G$. Of particular interest, and commonly encountered, are inverse problems that are \textit{ill-posed}. Ill-posedness means the following:

- **Existence**: There might not be any solution for $m$
- **Uniqueness**: There might be infinitely many solutions for $m$
- **Sensitivity to input**: The inverse solution $m$ is very sensitive to small changes in $d$

\textit{The basic goal of inverse methods is to constrain the model parameters $m$ such that the forward model $G$ fits the data $d$ to within acceptable errors. Fitting data exactly is not desirable!}

The following example is from a simple glacier flow model. The red crosses show measurements of surface velocities; ice thickness is assumed to be known. The cyan curve shows a possible solution for basal motion that fits the surface data exactly. This is not a reasonable solution for basal velocities.

There are two issues we will focus on:

- Don’t attempt to fit data better than what is warranted by the appropriate errors
- There is not ’a solution’ to the inverse problem, there are often many solutions and we need criteria to decide which solutions are more likely.
Methods for solving inverse problems

There are several methods for solving inverse problems. While the solution methods often seem very different, they have some commonalities: All of them use some additional criteria to constrain possible or likely solutions. Some methods are more explicit about these choices than others. We will shortly consider a few.

Norm minimization

We define an error function that describes how well the model fits the data:

\[ E^2 = \sum_{j=1}^{N} \frac{|d_j - G_j(m)|^2}{\sigma_j^2}, \]

where the summation extends over all the data points, and \( \sigma_j \) denote errors. We want to find model parameters \( m \), such that this error function is as small as should be expected from the errors in the data. That is, we fit the data to within some tolerance \( T \), which is defined through the statistics of the error distribution. But, in general, there are many solutions for \( m \), such that \( E^2 = T^2 \). To choose one of these solutions we optimize a certain property of \( m \). This property is expressed as a norm that can be minimized. So, we are looking for a solution that minimizes a certain measure of \( m \). We define a functional:

\[ U(\lambda, m) = ||m||_w - \lambda(E^2 - T^2) \]

We will find \( \lambda \) and \( m \), to minimize this functional \( U \). This is a constrained minimization problem, \( \lambda \) is a so-called Lagrange multiplier and \( ||.||_w \) is some sort of a norm that measure the size of \( m \) in some sense. Minimizing \( U \) amounts to minimizing this norm subject to the constraint \( E^2 = T^2 \). The solution to the problem can be found by requiring that \( \frac{\partial U}{\partial \lambda} = 0 \) and \( \frac{\partial U}{\partial m} = 0 \).

If the norm \( ||.||_w \) is the usual 2-norm, then this example will provide the smallest solution \( m \) that fits the data within the given tolerance. But the norm can also measure another property. A common choice is a so-called Sobolev norm:

\[ ||f||_s = \left( \gamma f^2(a) + \int_a^b (f'(x))^2 dx \right)^{1/2} \]

This norm is a measure of the roughness of a function. Minimizing \( U \) therefore amounts to finding a solution that minimizes roughness. The following example applies this to the same simple glacier model that was shown above. Again, the red crosses show surface velocity 'data'. The cyan curve fits these data to within a given tolerance, at the same time minimizing the roughness of the basal speed.
A very similar methodology is known among mathematicians as Tikhonov regularization. The point of view there is to use the norm as a way to ‘regularize’ the solution. The functional to minimize is then

\[ U(m) = ||m||_w - \lambda E^2 \]  

and \( \lambda \) is treated as a trade-off parameter that needs to be determined.

**Bayes method**

A popular method for solving inverse methods is based on Bayes’ Theorem. This is perhaps the clearest of all inverse methods techniques, because it emphasizes most clearly that there are many possible solutions. All quantities are treated like probability distributions \( \rho(\cdot) \):

\[ \rho(m|d) = \frac{\rho(m)\rho(d|m)}{\rho(d)} \]  

In this equation \( \rho(m|d) \) is the distribution of \( m \) given some data \( d \). This is called the **posterior**. \( \rho(m) \) is known as the **prior**, which describes the knowledge about \( m \) before the inverse method is applied. \( \rho(d|m) \) is the forward model (or likelihood), it gives the distribution of \( d \), given the parameters \( m \). Finally, \( \rho(d) \) describes the distribution of the data \( d \) under all possible parameter values. In practice, this is difficult to estimate and that factor is just used to normalize \( \rho(m|d) \).

In a nutshell, Bayes’ Theorem provides a methodology to derive likelihood functions for \( m \), given a physical model \( \rho(d|m) \) and data with errors, starting with some prior assumptions about \( m \). If little is known about \( m \), uniform prior distributions are used. But often, the parameters \( m \) are constrained by some prior knowledge. For example, physics might demands that \( m \) must satisfy certain bounds.

One way to implement Bayes’ method is to use Monte Carlo methods to sample the relevant distributions.

**Solution methods**

There are many numerical methods for solving inverse problems. They appear under names such as data assimilation, control methods, iterative methods, adjoint methods, L-curve
criteria, etc. All of these share some commonalities and generally follow some sort of norm minimization or Bayesian ideas. The most important take-home points for all inverse methods are:

- Your model must not fit data better than warranted by errors in the data and the model. If you do so, your results will be spurious and often entirely meaningless, implying a false precision (that’s what’s meant by the quote at the beginning).

- All inverse methods depend on some additional assumptions, which are not always clearly stated.

Note that inverse methods have other uses as well, that I haven’t discussed here. Most importantly, they can be used to optimally design a data acquisition strategy.

**Some examples in glaciology**

I will provide just three examples from the literature here. Of course, there are many, many more, but this shows a bit of the spectrum of applications.

**Finding past accumulation rates**


- Forward model: Given past accumulation rates and velocity fields, one can calculate the position of isochronous (equal age) layers in firn and ice

- Inverse model: Use radar layers to constrain the velocity field and accumulation rates

- Methodology: He is essentially using a norm minimization method, implemented through the Singular Value Decomposition

**Finding basal conditions from surface observations**


- Forward model: Given basal slipperiness, ice rheological parameters, and a flow model, surface velocities can be calculated

- Inverse model: Using surface measurements of velocity and thickness change, infer basal slipperiness and ice rheology

- Methodology: A norm minimization, using an iterative inverse technique until sufficient convergence is reached
Finding ice thickness


- Forward model: Given a flow model, ice thickness, and surface mass balance, velocity fields and thinning rates can be calculated
- Inverse model: Use constraints on and measurements of surface mass balance, thinning rates and surface velocities to determine ice thickness
- Methodology: Bayesian methods