Inverse methods in glaciology

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General Problem Setting

Formal problem statement

Solution methods

Outline

General Problem

Common situation in geophysics:

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- You have a certain understanding of the world that is expressed in a set of equations (forward model G)
- You would like to derive a set of parameters (model m)
- You would know how to get from m to d (forward model), but the reverse takes special treatment

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- Finding a brain tumor with a CAT scan

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- Finding past accumulation from radar layers
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- Finding initial conditions for ice sheet models given all available observations

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- We would like to go the other way, but G might not have a well-defined inverse
- Finding m from d is often an *ill-posed* problem

Ill-posed problems

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- The problem might have many solutions
- The solution might be badly defined, i.e. small changes in input lead to large changes in output
- Honest mathematicians keep their hands off such problems

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- For linear inverse problems useful theorems can be derived (such as existence of solutions, etc
- Non-linear problems are much more difficult. Often the methods involve linearization and iteration.

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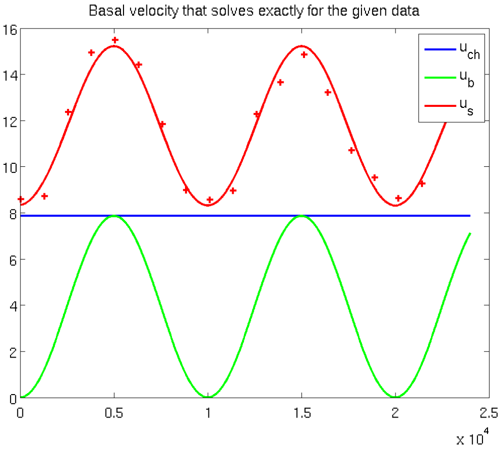
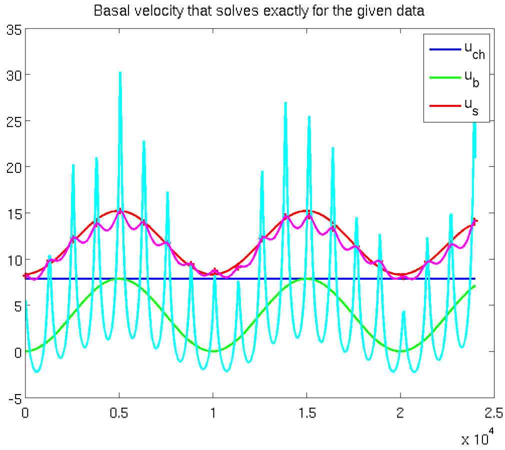
- The discretized problem has many solutions. How do you choose one?
- Minimize a property of the solution that can be expressed as a norm
- The choice of norm determines the solution (user input or a *prior*)

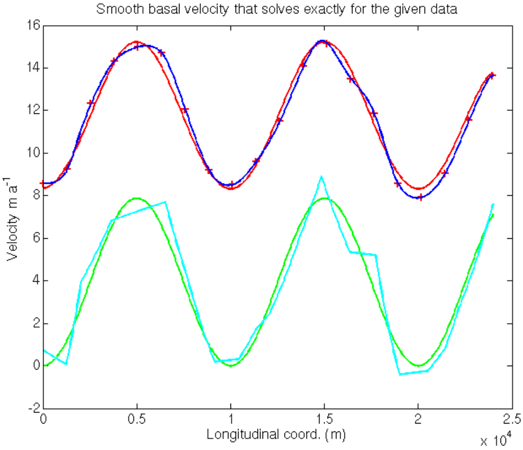
Generating data to be used in an example



Finding the smallest solution



Finding the smoothest solution



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- For example, the norm $\| f \|_s = \left(\gamma f^2(a) + \int_a^b (f'(x))^2 dx \right)^{1/2}$ penalizes roughness

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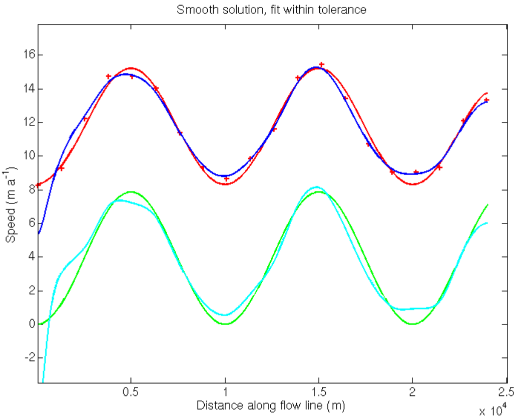
- λ is a Lagrange multiplier that can be solved for if T is known

Finding a solution within a tolerance



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- One makes a prior assumption about the model parameters $\rho(m)$
- Apply Bayes' Theorem: $\rho(m|d) = \frac{\rho(m)\rho(d|m)}{\rho(d)}$
- *The probability of m given d is equal to the prior assumption times the probability of d given m (the forward model) divided by the probability distribution of the data*

Iterative methods

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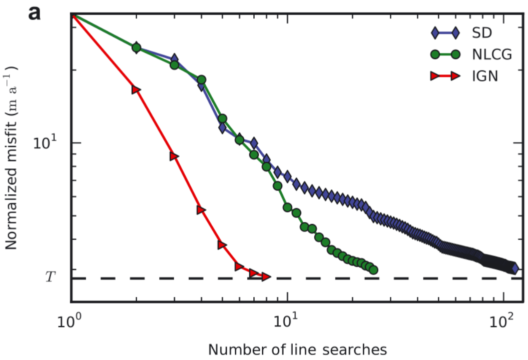
- Stop once the data are fit *well enough*

What is well enough?



The L-curve method

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- Each method finds a solution and not the solution
- A solution of the inverse problem is a set of model parameters that is consistent with the forward model and the data within errors
- Each method involves a number of assumptions