Inverse methods in glaciology

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General Problem Setting

Formal problem statement

Solution methods
Outline

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General Problem

Common situation in geophysics:

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Common situation in geophysics:
- You have observables (data \(d\))
- You have a certain understanding of the world that is expressed in a set of equations (forward model \(G\))
- You would like to derive a set of parameters (model \(m\))
- You would know how to get from \(m\) to \(d\) (forward model), but the reverse takes special treatment
Examples of inverse problems

- Finding the seismic velocity structure of the Earth from measurements of seismic arrival times
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- Finding oil by active seismics
- Finding a brain tumor with a CAT scan
Examples of inverse problems in glaciology

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- Finding past accumulation from radar layers
- Finding ice thickness from gravity anomalies
- Finding initial conditions for ice sheet models given all available observations
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The forward model consists of an equation or a set of equations that can calculate observables from model parameters: $G : m \rightarrow d$
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Forward model

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- We would like to go the other way, but $G$ might not have a well-defined inverse
- Finding $m$ from $d$ is often an *ill-posed* problem
Ill-posed problems

- The problem might not have a solution
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- The problem might have many solutions
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- The solution might be badly defined, i.e. small changes in input lead to large changes in output
Ill-posed problems

- The problem might not have a solution
- The problem might have many solutions
- The solution might be badly defined, i.e. small changes in input lead to large changes in output
- Honest mathematicians keep their hands off such problems
Linear inverse problems

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- For linear inverse problems useful theorems can be derived (such as existence of solutions, etc)
- Non-linear problems are much more difficult. Often the methods involve linearization and iteration.
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- The discretized problem has many solutions. How do you choose one?
- Minimize a property of the solution that can be expressed as a norm
- The choice of norm determines the solution (user input or a prior)
Generating data to be used in an example

Basal velocity that solves exactly for the given data

- $u_{ch}$
- $u_b$
- $u_s$

$x \times 10^4$

0 0.5 1 1.5 2 2.5
Finding the smallest solution

Basal velocity that solves exactly for the given data

Solution methods
Finding the smoothest solution

Smooth basal velocity that solves exactly for the given data

Velocity m a⁻¹

Longitudinal coord. (m)

x 10^4
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\[ \|f\|_s = \left( \gamma f^2(a) + b \int_a^x (f')^2 \, dx \right)^{1/2} \]

penalizes roughness
Choice of norm

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- The only difference is the choice of norm that is minimized to select among all possible solutions.
- For example, the norm $\| f \|_s = \left( \gamma f^2(a) + \int_a^b (f'(x))^2 \, dx \right)^{1/2}$ penalizes roughness.
Accounting for errors

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- \( \lambda \) is a Lagrange multiplier that can be solved for if \( T \) is known
Finding a solution within a tolerance
Bayesian methods

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- Ex: Observations with random errors $\rho(d) \propto \exp\left(-(d/\sigma)^2\right)$
- One makes a prior assumption about the model parameters $\rho(m)$
- Apply Bayes’ Theorem: $\rho(m|d) = \frac{\rho(m)\rho(d|m)}{\rho(d)}$
- The probability of $m$ given $d$ is equal to the prior assumption times the probability of $d$ given $m$ (the forward model) divided by the probability distribution of the data
Iterative methods

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Iterative methods

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- Calculate the misfit to observations
- Calculate a correction to lower the misfit
- Stop once the data are fit *well enough*
What is well enough?

The L-curve method
Commonalities of inverse methods

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- Each method finds a solution and not the solution
- A solution of the inverse problem is a set of model parameters that is consistent with the forward model and the data within errors
- Each method involves a number of assumptions