# CONTINUUM MECHANICS

Martin Truffer
University of Alaska Fairbanks

2022 McCarthy Summer School

# Contents

# 1

# Introduction

Continuum mechanics is the application of classical mechanics to continous media. So, well first look at:

- What is Classical mechanics?
- What are continuous media?

## 1.1 Classical mechanics: a very quick summary

We make the distinction of two types of equations in classical mechanics: (1) Statements of conservation that are very fundamental to physics, and (2) Statements of material behavior that are only somewhat fundamental and contain empirical parameters that are specific to the material in question.

### Conservation Laws

Statements of physical conservation laws:

- Conservation of mass
- Conservation of linear momentum (Newton's Second Law)
- Conservation of angular momentum

- Conservation of energy

There are other conservation laws (such as those of electric charge), but these are of no further concern to us right now.
Conservation Laws are good laws. Few sane people would seriously question them. If your theory/model/measurement does not conserve mass or energy, you have most likely not discovered a flaw with fundamental physics, but rather, you should doubt your theory/model/measurement. In fact, conservation laws provide very good tests, for example for numerical models.
We will see that conservation laws are not enough to fully describe a deforming material. Simply said, there are fewer equations than unknowns. We also need equations describing material behavior.

## Material (constitutive) laws

Material or constitutive laws describe the reaction of a material, such as ice, to forcings, such as stresses, temperature gradients, increase in internal energy, application of electric or magnetic fields, etc. Such 'laws' are often empirical (derived from observations rather than fundamental principles) and involve material-dependent 'constants'. Examples are:

- Flow law (how does ice deform when stressed?)
- Fourier's Law of heat conduction (how much energy is transfered across a body of ice, if a temperature difference is applied?)

There are other examples that we will not worry about here.
Constitutive laws are not entirely empirical. They have to be such that they don't violate basic physical principles. Perhaps the most relevant physical principle here is the *Second Law of Thermodynamics*. The material laws have to be constrained, so that heat cannot spontaneously flow from cold to hot, or heat cannot be turned entirely into mechanical work. There is a long (and complicated!) formalism associated with that; we will not be further concerned with it.
There are other requirements for material laws. The behavior of a material should not change if the coordinate system is changed (*material objectivity*), and any symmetries of the material should be considered. For example, the ice crystal's hexagonal structure implies certain symmetries that ought to

be reflected in a flow law. Here we assume that ice is isotropic (looks the same from all directions). This assumption is based on the observation that often (but not always) ice grains occur randomly oriented in glaciers. But this is not always a good assumption.
Conservation laws and constitutive laws constitute the *field equations*. The field equations together with boundary conditions form a set of partial differential equations that solve for all the relevant variables (velocity, pressure, temperature) in an ice mass. The goal here is to show how we get to these field equations.

## 1.2 Continuous media

### Densities

Classical mechanics has the concept of point mass. We attribute a finite mass to an infinitely small point. We track the position of the point and by looking at rates of change of position we determine velocity and then acceleration. This is known as *kinematics*. We then assess all the forces that affect a point mass or a collection of them. Newton's 2nd Law ($F = ma$) then determines accelerations (that's *dynamics*).
Ice forms a finite sized body of deformable material (a fluid). The challenge then is to write the laws for point masses such that they apply to continuous media. To define quantities at a point we introduce the concept of density. To introduce the density $\rho$, we acknowledge that some volume $\Omega$ of a fluid has a certain mass $m$. We then write:

$$m = \int_\Omega \rho dv \tag{1.1}$$

We can use this to define linear momentum:

$$mv = \int_\Omega \rho v dv \tag{1.2}$$

and internal energy

$$U = \int_\Omega \rho u dv \tag{1.3}$$

We define these quantities somewhat carelessly. In particular, the concept of density and the mathematical methods of continuum mechanics imply a

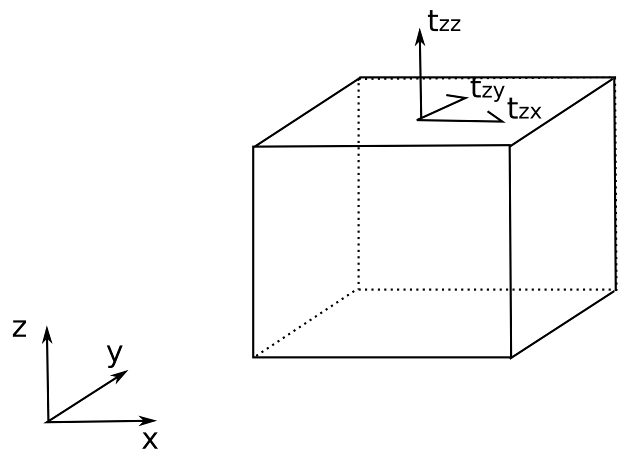


Figure 1.1: A force applied to the face of a representative volume can be decomposed into three components

mathematical limit process to infinitely small volumes (a point). This does not make immediate physical sense, as the physical version of this limit process would go from ice sheet scale to individual grains, then molecules, atoms, atomic structure, etc. This would eventually involve physics that is quite different from classical physics. We will assume that the concepts of continuum mechanics apply to volumes that are 'sufficiently small' here, and will again not worry about the details.

## Oh no, tensors!

The description of continuous media requires the introduction of a new mathematical creature, the tensor. This is needed to describe forces in continuous media. Let's cut a little cube out of an ice sheet and try to see in how many ways we can apply forces to it (see Figure 1.1).
A representative little cube has six faces. Each face can be described by a surface normal vector, and each face can be subject to a force. A force is a vector quantity, so it has three components. We choose one component along the surface normal and define it as positive for tension and negative for compression. The other two directions are tangential to the face and perpendicular to each other. Those are shear forces. In analogy to the definition of densities, we now define stresses as forces per unit area. So for each face we end up with three stresses.

Because there are so many faces and force directions we have to agree on a notation. The stress acting on a face with surface normal $i$ and in the direction $j$ is written as $t_{ij}$.
There are three spatial directions ($x$, $y$, $z$) and each one of them has a three component force vector associated with it. This leaves us with nine components of the stress tensor. These nine components are usually ordered as follows:

$$\mathbf{t} = \begin{bmatrix} t_{xx} & t_{xy} & t_{xz} \\ t_{yx} & t_{yy} & t_{yz} \\ t_{zx} & t_{zy} & t_{zz} \end{bmatrix} \tag{1.4}$$

$\mathbf{t}$ is known as the *Cauchy Stress Tensor*.
A tensor is not just any table of nine numbers. It has some very special properties that relate to how it changes under a coordinate transformation. A rotation in 3D can be described by an orthogonal matrix $R$ with the properties $R^{-1} = R^T$ and $\det R = 1$. A second order tensor transforms under such a rotation as

$$\mathbf{t}' = \mathbf{R}\mathbf{t}\mathbf{R}^T \tag{1.5}$$

The way to think about this is that two rotations are involved in this transformation, one of the face normal, and one of the force vector.
Tensors have quantities associated with them that are invariant under transformations. A second order tensor has three such invariants: Invariants

$$\mathrm{I}_{\mathbf{t}} = \mathrm{Tr}\,\mathbf{t} \tag{1.6}$$

$$\mathrm{II}_{\mathbf{t}} = \frac{1}{2}\left(\mathrm{Tr}\,\mathbf{t^2} - (\mathrm{Tr}(\mathbf{t}))^2\right) \tag{1.7}$$

$$\mathrm{III}_{\mathbf{t}} = \det(\mathbf{t}) \tag{1.8}$$

Here, Tr refers to the trace ($\mathrm{Tr}\,(\mathbf{t}) = t_{xx} + t_{yy} + t_{zz}$) and det to the determinant.
Remember the principle of material objectivity? Tensor invariants are interesting quantities for finding material laws, because they do not change with a change of coordinate system. For example, a flow law that depends explicitly on the stress component $t_{xz}$ violates material objectivity, because that component changes under coordinate transformations. But, a flow law

that depends on $\mathrm{II}_{\mathbf{t}}$ is fine, because the second invariant does not change under coordinate transformations.
Other important tensors are the strain tensor $\varepsilon$ and the strain rate tensor $\dot{\varepsilon}$ or $D$. The strain tensor is important for elastic materials. While ice is elastic at short time scales, we will be mainly concerned with the viscous deformation of ice. The relevant quantity is then the strain rate tensor. Its components are defined by

Strain rate tensor

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.9}$$

Here, $u_i$ are the velocity components and $x_i$ are the spatial coordinates.

## A little bit on notation

Notation conventions in continuum mechanics vary greatly. It is not uncommon to see $\nabla \cdot \mathbf{v}$, $\operatorname{div} \mathbf{v}$, $v_{i,i}$, or $\partial_i v_i$ for the same quantity. We will introduce the last quantity here. While it might not be as familiar looking as the others, it greatly simplifies calculations when second order tensors are involved.
For the coordinates of a point $\mathbf{x}$ we use $(x, y, z)$ interchangeably with $(x_1, x_2, x_3)$, and for velocity $\mathbf{v}$ we use $(u, v, w)$ or $(u_1, u_2, u_3)$.
When we deal with tensors of first and second rank and with derivatives, the standard notation can quickly become very awkward. We therefore introduce the following conventions:

- Repeating indices indicate summation. This is known as the *Einstein convention.* For example, $\mathrm{Tr}\,\mathbf{t} = t_{xx} + t_{yy} + t_{zz} = t_{ii}$
- $\partial_i$ is used for differentiation with respect to $x_i$. For example, $\partial_j u_i = \frac{\partial u_i}{\partial x_j}$

Some other examples include:

- Strain rate tensor $D_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$
- Divergence of a vector: $\nabla \cdot \mathbf{v} = \partial_i v_i$
- $i$-th component of the gradient of a scalar: $(\nabla s)_i = \partial_i s$
- Scalar product: $\mathbf{u} \cdot \mathbf{v} = u_i v_i$

# 2

# Field equations for ice flow

## 2.1 Conservation Laws

We will find mathematical expressions that express the conservation of mass, linear momentum, angular momentum, and energy. We will accomplish this by first formulating a general conservation law.

### General conservation laws

Imagine a volume $\Omega$ of ice enclosed by a boundary $\partial\Omega$. Now imagine some quantity $\Psi$ with density $\psi$ contained in that volume ($\Psi$ will be mass, momentum, and energy). So we can write

$$\Psi(t) = \int_\Omega \psi(\mathbf{x}, t) dv \tag{2.1}$$

It is a simple consideration that this quantity $\Psi$ can only change in two ways: Either there is a supply $S$ within $\Omega$ or there is a flux $F$ of the quantity through its boundary $\partial\Omega$. We assume that $\Omega$ does not depend on time.
(Note: sometimes a distinction is made between 'supply' and 'production'. We will not be concerned with this here.)
We assume that the supply $S$ also has an associated density $s$, so that we can write

$$S(t) = \int_\Omega s(\mathbf{x}, t) dv \tag{2.2}$$

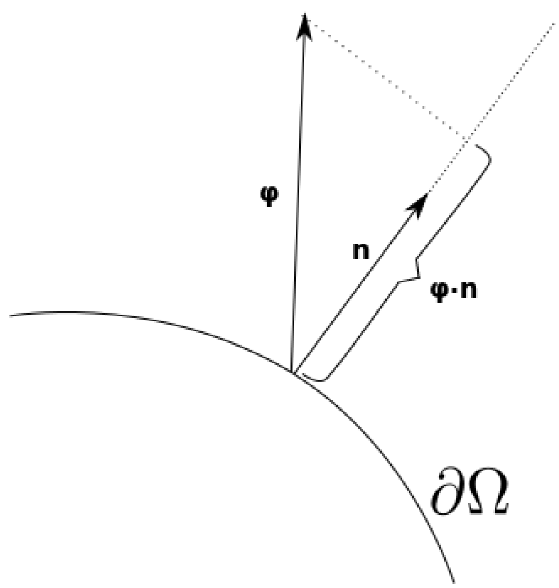


Figure 2.1: The component of a flux vector $\varphi$ that is directed in or out of a surface $\partial\Omega$ is given by $\boldsymbol{\varphi} \cdot \mathbf{n}$. Note the sign convention.

If we think of $\Omega$ as independent of time, then the flux $F$ across a boundary can have more than one contribution. A first contribution is the amount of the quantity $\psi$ that is being carried across the boundary with the velocity field $\mathbf{v}$. It is given by $\psi\mathbf{v}$. There can be other fluxes, which for now we will designate by $\boldsymbol{\varphi}$:

$$F(t) = \oint_{\partial\Omega} (\psi\mathbf{v} + \boldsymbol{\varphi}(\mathbf{x}, t)) \cdot \mathbf{n} da \tag{2.3}$$

$\boldsymbol{\varphi}$ is the flux density, and $\mathbf{n}$ the local normal pointing vector to the surface. It is not immediately obvious that eqn. 2.3 can be written that way, but it does make some sense, because only the component of the flux perpendicular to a surface contributes to in- or outflow (Fig. 2.1).

We can now formulate a general balance law:

$$\begin{aligned} \frac{d\Psi}{dt} &= S - F \qquad (2.4) \\ \frac{d}{dt}\int_\Omega \psi(t)dv &= \int_\Omega s(t)dv - \oint_{\partial\Omega} (\psi\mathbf{v} + \boldsymbol{\varphi}) \cdot \mathbf{n} da \qquad (2.5) \end{aligned}$$

Because we chose $\Omega$ such that it is a fixed volume in space, i.e. $\Omega \neq \text{fct}(t)$, the time derivative can be carried inside the integral. A further simplification is reached by invoking Gauss' Theorem:

$$\oint_{\partial\Omega} (\psi\mathbf{v} + \boldsymbol{\varphi}) \cdot \mathbf{n} da = \int_\Omega \nabla \cdot (\psi\mathbf{v} + \boldsymbol{\varphi}) dv \qquad (2.6)$$

A general balance law is thus

$$\int_\Omega \frac{\partial\psi}{\partial t} dv = \int_\Omega s(t)dv - \int_\Omega \nabla \cdot (\psi\mathbf{v} + \boldsymbol{\varphi}) dv \qquad (2.7)$$

When we derived this law, we made no assumptions about the shape and size of $\Omega$. In particular, we can make it infinitely small. This integral form of the balance law then reduces to its local form

$$\frac{\partial\psi}{\partial t} = s(t) - \partial_i(\psi v_i + \varphi_i) \qquad (2.8)$$

where we have now used the notation introduced above.
All that remains now is to identify the relevant terms for $\psi$, $s$, and $\boldsymbol{\varphi}$.

## Conservation of mass

In the case of mass we have $\psi = \rho$, $s = 0$, and $\boldsymbol{\varphi} = 0$. This gives us the mass conservation law

$$\frac{\partial\rho}{\partial t} = -\nabla \cdot (\rho\mathbf{v}) = -\partial_i(\rho v_i) \qquad (2.9)$$

This equation is greatly simplified by the fact that the density of ice is constant; ice is a so-called *incompressible* material:

$$\nabla \cdot \mathbf{v} = \partial_i v_i = 0 \qquad (2.10)$$

## Conservation of momentum

Momentum is a vector quantity (or first rank tensor). Its density is given by $\boldsymbol{\psi} = \rho\mathbf{v}$. There is a supply of momentum within any given volume, namely that of gravity: $\mathbf{s} = \rho\mathbf{g}$. There is also a surface boundary flux of momentum into $\Omega$, which is provided by surface stresses. Think of Newton's Second Law: Forces (stresses) are a source of momentum.
The flux term is then $\boldsymbol{\varphi} = -\mathbf{t}$, i.e. the Cauchy stress tensor. Note the negative sign and the fact that $\boldsymbol{\varphi}$ is now a second rank tensor. This produces a momentum balance of

$$\frac{\partial \rho v_i}{\partial t} = -\partial_j(\rho v_i v_j) + \partial_j t_{ij} + \rho g_i \tag{2.11}$$

Using the product rule for the left hand side and for

$$\partial_j(\rho v_i v_j) = (\partial_j v_i)\rho v_j + v_i \partial_j(\rho v_j) \tag{2.12}$$

The second term equals $v_i \frac{\partial \rho}{\partial t}$ due to mass conservation (eqn. 2.9) and we are left with:

$$\rho \frac{\partial v_i}{\partial t} + \rho(\partial_j v_i)v_j = \partial_j t_{ij} + \rho g_i \tag{2.13}$$

The left hand side is often written as

$$\rho \frac{dv_i}{dt} = \rho \frac{\partial v_i}{\partial t} + \rho(\partial_j v_i)v_j \tag{2.14}$$

The symbol $\frac{d}{dt}$ denotes the *total derivative*. It is instructive to think about this in general terms: the change of a quantity at one point is due to changes in time at that location ($\frac{\partial}{\partial t}$) plus whatever is carried there from 'upstream', which is a product of the velocity with the gradient of that quantity.
In glaciology we simplify eqn. 2.14 further by neglecting accelerations. Using typical numbers for ice flow (even very fast flow), it can be shown that $\rho\frac{d\mathbf{v}}{dt}$ is always much smaller than the other terms in eqn. 2.14. This approximation is known as *Stokes Flow* and is typical for creeping media. We now have:

$$\partial_j t_{ij} + \rho g_i = 0 \tag{2.15}$$

You will sometimes encounter this equation in the following notation:

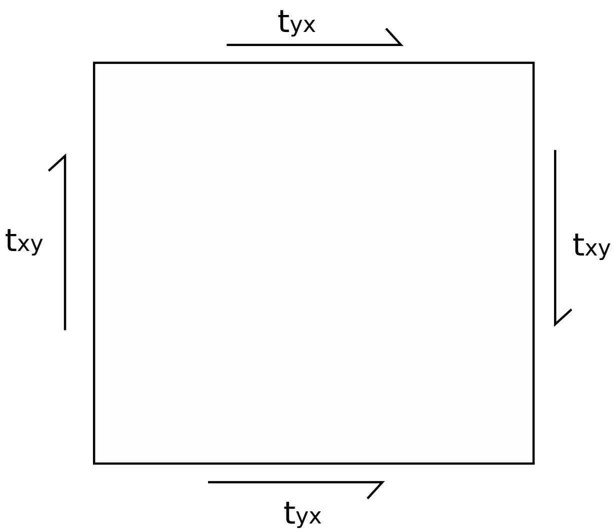


Figure 2.2: If $t_{ij} \neq t_{ji}$ a net torque and angular acceleration would result.

$$\nabla \cdot \mathbf{t} + \rho \mathbf{g} = 0 \tag{2.16}$$

## 2.2 Conservation of angular momentum

Conservation of angular momentum results in a complicated expression that can be greatly simplified to yield

$$t_{ij} = t_{ji} \tag{2.17}$$

An intuitive way of illustrating this is figure 2.2. If the stress tensor were not symmetric, a net torque would result that would lead to angular acceleration.

A symmetric stress tensor has the interesting property that there is always an orthogonal transformation that diagonalizes the tensor. In other words, one can always find an appropriately oriented coordinate system in which no shear stresses occur. The stresses along the main axes of such a coordinate system are known as *principal stresses*. This can be useful for finding maximum tensional stresses, which determine the direction of crevassing.

## 2.3 Conservation of energy

The energy density is given by

$$\psi = \rho(u + \frac{v^2}{2}) \tag{2.18}$$

where $u$ is the internal energy per unit mass and $v^2 = v_i v_i$. The first term is the inner energy, while the second one is the kinetic energy. There is a supply of energy, which is given by the work done by gravity:

$$s = g_i v_i \tag{2.19}$$

Finally, there are two flux terms, one is the heat flux $\mathbf{q}$, the other one is the frictional heat due to stresses (i.e. the work done by the stresses):

$$\varphi_i = q_i - t_{ij} v_j \tag{2.20}$$

Note the opposite signs: A positive heat flux implies that heat is carried away from our sample volume, while a positive work term for the surface stresses results in heat supplied to the sample volume.
We thus obtain an energy balance equation

$$\frac{\partial}{\partial t}\left[\rho\left(u + \frac{v^2}{2}\right)\right] = -\partial_i\left[\rho\left(u + \frac{v^2}{2}\right) v_i\right] - \partial_i q_i + \partial_i(t_{ij} v_j) + \rho g_i v_i \tag{2.21}$$

We note, using the momentum balance (eqn. 2.11) multiplied with $v_i$ that

$$\frac{\partial}{\partial t}\left[\rho \frac{v^2}{2}\right] + \partial_i\left[\rho \frac{v^2}{2} v_i\right] = \rho v_i \frac{dv_i}{dt} = \partial_j(t_{ij}) v_i + \rho g_i v_i \tag{2.22}$$

Note that this holds even without the Stokes approximation. We can use the product rule to get

$$\partial_i(t_{ij} v_j) = \partial_i(t_{ij}) v_j + t_{ij} \partial_i v_j = \partial_j(t_{ij}) v_i + t_{ij} \partial_j v_i \tag{2.23}$$

The second equality follows from the symmetry of $t_{ij}$. We also note that $t_{ij}\partial_j v_i = t_{ij}\partial_i v_j$, so that

$$t_{ij} D_{ij} = \frac{1}{2}(t_{ij}\partial_j v_i + t_{ij}\partial_i v_j) \tag{2.24}$$

where $D_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$ is the strain rate tensor.
This leaves us with the following equation for energy conservation.

$$\rho \frac{du}{dt} = -\partial_i q_i + t_{ij} D_{ij} \tag{2.25}$$

or

$$\rho \frac{du}{dt} = -\nabla \cdot \mathbf{q} + \mathrm{Tr}(\mathbf{tD}) \tag{2.26}$$

## 2.4 Summary of conservation equations

We can now summarize what we have learned from the conservation of mass, linear and angular momentum, and energy for an incompressible Stokes fluid. We present the equations in comma notation with Einstein summation

$$\partial_i v_i = 0 \tag{2.27}$$

$$\partial_j t_{ij} + \rho g_i = 0 \tag{2.28}$$

$$t_{ij} = t_{ji} \tag{2.29}$$

$$\rho \frac{du}{dt} + \partial_i q_i - t_{ij} D_{ij} = 0 \tag{2.30}$$

as well as the, perhaps, more familiar form

$$\nabla \cdot \mathbf{v} = 0 \tag{2.31}$$

$$\nabla \cdot \mathbf{t} + \rho \mathbf{g} = 0 \tag{2.32}$$

$$\mathbf{t} = \mathbf{t}^T \tag{2.33}$$

$$\rho \frac{du}{dt} + \nabla \cdot \mathbf{q} - \mathrm{tr}(\mathbf{tD}) = 0 \tag{2.34}$$

These present a total of 5 equations (not counting 2.30). Unfortunately, we 13 unknowns (3 velocity components, 6 elements of the symmetric stress tensor, the internal energy, and three components of the heat flux), so additional equations are needed. These are equations that describe the material behavior of ice, so we'll look at them next.

## 2.5 Constitutive relations

### Viscous flow

Stressed ice can have a variety of responses, depending on the magnitude of stress and the time scales involved. Possible responses involve brittle fracture, elastic recoverable deformation, and viscous (non-recoverable) deformation. We will restrict our considerations to viscous deformation.
It has been found experimentally that the application of a shear stress $\tau$ will result in deformation

$$\dot{\varepsilon} = A\tau^n \tag{2.35}$$

where $\dot{\varepsilon}$ is the strain rate, $A$ is a flow-rate factor (which is strongly temperature dependent), and $n$ is an exponent, often assumed to be 3. This is known as a *Glen-Steinemann* flow law among glaciologists, but it turns out to be quite common for describing the deformation of other solids, such as metals. The relation is non-linear, the ice gets softer at higher stresses. It is also common to write this in terms of viscosity $\eta$:

$$\dot{\varepsilon} = \frac{1}{2\eta}\tau \tag{2.36}$$

For ice, the viscosity can then be written as

$$\eta = \frac{1}{2A\tau^{n-1}} \tag{2.37}$$

This clearly shows that the viscosity is stress dependent, and becomes lower at higher stresses. It also shows the peculiarity of infinite viscosities at zero stresses. There are good theoretical and experimental reasons why this should not be so, and $\eta$ is often modified to account for that.
Eqn. 2.35 relates one stress component to one strain rate component. But the law can be generalized to account for the full stress state as given by the stress tensor. To do this requires the realization, however, that a uniform pressure cannot lead to deformation in an incompressible material. We therefore have to define a new tensor, called the *deviatoric stress tensor* $\mathbf{t}'$, that indicates the departure from a mean pressure $p$:

$$t_{ij} = \frac{1}{3}t_{kk}\delta_{ij} + t'_{ij} = -p\delta_{ij} + t'_{ij} \tag{2.38}$$

where $\delta_{ij}$ is the *Kronecker symbol*. Its value is 1 if $i = j$, and 0 otherwise. We have now introduced a new variable, the pressure $p = -\frac{1}{3}t_{kk}$. The deviatoric stress tensor is thus traceless ($t'_{ii} = 0$), by definition, and is the relevant quantity for ice deformation. A possible generalization for eqn. 2.35 is the *Glen-Nye* flow law:

$$D_{ij} = A(T)\mathrm{II}_{t'}^{\frac{n-1}{2}} t'_{ij} \tag{2.39}$$

$\mathrm{II}_{t'}$ is the second invariant of the stress deviator (in older literature also known as the octahedral stress):

$$\mathrm{II}_{\mathbf{t}'} = \frac{1}{2}\left(\mathrm{tr}(\mathbf{t}'^2) - (\mathrm{tr}\,\mathbf{t}')^2\right) = \frac{1}{2}(\mathrm{tr}\,\mathbf{t}'^2) \tag{2.40}$$

Note, that $t'_{kk}$ and $D_{kk}$ both vanish, i.e. both tensors are traceless.
It is an easy exercise to show that eqn. 2.39 reduces to eqn. 2.35 in the presence of only one stress component.
The flow rate factor $A$ is strongly dependent on temperature via an *Arrhenius relationship*:

$$A(T) = A_0 e^{-\frac{Q}{kT}} \tag{2.41}$$

where $Q$ is an activation energy, and $k$ is the Boltzmann constant. The value of $Q$ must be determined experimentally, and it appears to change value for temperatures greater than $-10°$ C.
Note that many glaciers are at or very close to the pressure-dependent melting point, so that the temperature is known. In that case, the mass and momentum balance together with the flow law now form 10 equations for the 10 unknowns (3 velocity components, pressure, and 6 deviatoric stress components). With the appropriate boundary conditions we now have a solvable set of equations. It is common to invert equation 2.39 and then replace the stress tensor in the momentum balance. This leads to the *Navier-Stokes equations* for non-linear creeping flow.
Also note that there is a fundamental difference between the flow law discussed in this section and the conservation laws in the previous section. The conservation laws are based on fundamental physics. The flow law is based on a series of experiments and some theory. For example, one has to determine experimentally whether the third invariant should also enter eqn. 2.39, or what the values of $A_0$, $Q$, and $n$ are. There are also other dependencies for $A$, such as grain size, dust content, water content, etc.

If your carefully designed experiment shows a discrepancy with one of the conservation laws, you should be worried about the design of your experiment. Should it show a discrepency with the flow law, you might have good reason to be worried about the flow law.

### Cold ice

In cold ice, temperature enters as an additional variable. The additional equation to be solved is the energy equation. But it is again necessary to introduce two material relationships, one for the inner energy $u$ and one for the heat flow $\mathbf{q}$.

$$u = C_p T \tag{2.42}$$

defines the specific heat $C_p$, which is a measure of how much heat is needed to raise the temperature of a material by a certain temperature. Heat flow can be written in terms of *Fourier's Law*:

$$q_i = -k\partial_i T \tag{2.43}$$

where $k$ is the thermal conductivity. This reduces the energy equation to one in temperature only:

$$\rho\frac{d}{dt}(C_p T) - \partial_i(kT) - t_{ij}D_{ij} = 0 \tag{2.44}$$

The presence of $D_{ij}$ in this equation and the temperature dependence of the flow rate factor provide thermo-mechanical coupling for the field equations.

## 2.6 Boundary conditions

### Glacier surface

The glacier surface is subject to the atmospheric pressure

$$\mathbf{ntn} = -p_{atm} \tag{2.45}$$

A second condition describes the effects of climate (ablation/accumulation). It is necessary to recognize that the surface of a glacier is not a *material surface*. That is, a given set of ice particles that constitute the surface of the ice at time $t$, will, generally, not do so at any other times. This is because they will either be buried by additional accumulation, or melted. Also, the surface of the ice $z_s$ can move and does not need to be constant in time. The boundary condition is:

$$\frac{\partial z_s}{\partial t}|_{z_s=z_{\mathrm{surf}}(t)} = a + w \tag{2.46}$$

where $w$ is the vertical velocity component and $a$ is the accumulation/ablation function, which describes the amount of ice added or removed per unit time. This equation can also be written in terms of the divergence of the horizontal flux by integrating the divergence free condition vertically and replacing $w$:

$$\frac{\partial z_s}{\partial t}|_{z_s=z_{\mathrm{surf}}(t)} = a - \nabla_{xy} \cdot \mathbf{q} \tag{2.47}$$

where $\nabla_{xy}$ is the map-plane divergence and $\mathbf{q}$ is the integrated flux.
It is interesting to note that this is the only place where time occurs explicitly. The ice flow equations are *steady state equations* due to the Stokes approximation. The only time dependence enters through the surface kinematic equation. The equations are therefore sometimes referred to as *quasi steady-state*.
There is a second, hidden, possible time-dependence in the basal boundary condition due to the variability of basal water pressure.

## Base of the ice

There are several possible boundary conditions for the base of the ice. Generally, this is one of the most difficult topics of glaciology, as the base of the ice is not very amenable to observation.
For a frozen base, the boundary conditions are (seemingly) simple:

$$\mathbf{v}|_{z=z_{bed}} = 0 \tag{2.48}$$

There is, however, observational evidence for non-zero basal motion at a frozen bed, which is almost entirely ignored in the modeling world, because it remains well within other uncertainties.
In the case of a base at the melting point we first have to make sure that the bed-normal velocity matches the melt rate $\dot{m}$:

$$v_i n_i = \dot{m} \tag{2.49}$$

where $\mathbf{n}$ is the unit normal vector to the bed.
There are a variety of possible laws for basal motion. Most of them require a knowledge of the bed-parallel stress. If $\mathbf{t}$ is the stress tensor, then $\mathbf{tn}$ is the stress on a plane with surface normal $\mathbf{n}$. The bed-perpendicular component is then

$$(\mathbf{tn}) \cdot \mathbf{n} = t_{ij} n_j n_i = \mathbf{ntn} \tag{2.50}$$

and the bed-parallel component therefore

$$\tau_b = \mathbf{tn} - (\mathbf{ntn})\mathbf{n} \tag{2.51}$$

A common sliding law that has some theoretical justification is

$$\mathbf{v} = C\tau_b = C(\mathbf{tn} - (\mathbf{ntn})\mathbf{n}) \tag{2.52}$$

where $C$ is the slipperiness and can be a function of bed-roughness and water pressure.
If ice is underlain by till, theoretical and experimental evidence suggests a plastic boundary condition:

$$v_i = 0 \quad \text{if } |\tau_b| < \tau_{\text{yield}} \tag{2.53}$$

$$\tau_{bi} = \tau_{\text{yield}} \frac{v_i}{|v_i|} \quad \text{otherwise} \tag{2.54}$$

$\tau_{\text{yield}}$ is a till yield strength. Subglacial till does not deform if the applied stress is below the yield strength. Once the yield strength is reached, the sediment can deform at any rate. This is the characteristics of a friction law, or a perfectly plastic material. The yield strength depends on the difference between the pressure of the ice and the basal water pressure, as well as material properties of the till (given by a till friction angle).
If temperature is also modelled, it is common to prescribe the geothermal heat flux at the base of the ice, and using any excess heat for basal melt. Things get more complicated, because ice, upon reaching the melting point, becomes a mixture of liquid water and ice, and needs to be treated in proper mixture theory (see lecture by A. Aschwanden).

## Calving glaciers

A third type of boundary condition can arise where ice meets water (either ocean or lake). There is no generally agreed on calving rule that describes the process well. This is an important topic in glaciology, as many of the large observed changes in glaciers originate at the ice-water interface. Generally, a distinction is made between submarine melt (which depends on sea water temperature and subglacial discharge) and mechanical calving. The sum of the two is the total ice loss at the front, known as frontal ablation. But that's a topic of another lecture.