

# Thermodynamics of Glaciers

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## Solutions to Exercise

### 1 Climate history

The initial amplitude is  $\Delta T_0 = 2^\circ \text{C}$  during an oscillation period of 50 years, thus  $\omega = \frac{2\pi}{40 \text{ a}} = 3.9821 \times 10^{-9} \text{ s}^{-1}$ . With Equation (11) we get:

$$\frac{\Delta T(h)}{\Delta T_0} = \exp\left(-h\sqrt{\frac{\omega}{2\kappa}}\right).$$

The thermal diffusivity can be obtained from Equations (2) and (3)

$$\kappa(-3)^\circ \text{C} = \frac{k(-3^\circ \text{C})}{\rho c(-3^\circ \text{C})} = \frac{2.1 \text{ W m}^{-1} \text{ K}^{-1}}{910 \text{ kg m}^{-3} \cdot 2105.7 \text{ J kg}^{-1} \text{ K}^{-1}} = 1.34 \times 10^{-6} \text{ m}^2 \text{ s}^{-1},$$

or, just the value given in the script,  $1.09 \cdot 10^{-6} \text{ m}^2 \text{ s}^{-1}$  (the result then may slightly vary). Therefore we get

$$h = -\sqrt{\frac{2\kappa}{\omega}} \ln\left(\frac{\Delta T(h)}{\Delta T_0}\right) = -\sqrt{\frac{2 \cdot 1.34 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}}{3.9821 \times 10^{-9} \text{ s}^{-1}}} \ln\left(\frac{0.01}{2}\right) \approx 137 \text{ m}.$$

### 2 Melting temperature depression

From Figure (9) we see that the pressure melting temperature is  $T_m = -0.34^\circ \text{C}$ . We have  $p = 910 \text{ kg m}^{-3} \cdot 9.81 \text{ m s}^{-2} \cdot 300 \text{ m} + 75'000 \text{ Pa} = 2753130 \text{ Pa}$ . Using Equation (14) we obtain

$$\gamma = -\frac{T_m - T_{tp}}{p - p_{tp}} = \frac{273.15 \text{ K} - 0.34 \text{ K} - 273.16 \text{ K}}{2753130 \text{ Pa} - 611.73 \text{ Pa}} = 1.202 \times 10^{-7} \text{ K Pa}^{-1}.$$

In the literature we find the values  $\gamma = 7.42 \times 10^{-8} \text{ K Pa}^{-1}$  for pure ice and air-free water and  $\gamma = 9.8 \times 10^{-7} \text{ K Pa}^{-1}$  for pure ice and air-saturated water. The value calculated for Gornergletscher is even higher than  $\gamma$  for air-saturated water which means we have ice with air-saturated water.

### 3 Lake Vostok

1. Advection and diffusion
2. The Péclet number  $Pe$  is a measure of the relative importance of advection and diffusion.

3. First we calculate the pressure melting point at the base

$$\begin{aligned} T_m &= T_{tp} - \gamma(p - p_{tp}) \\ &= 273.16 \text{ K} - 7.42 \times 10^{-8} \text{ K Pa}^{-1} (910 \text{ kg m}^3 \cdot 9.81 \text{ m s}^{-2} \cdot 3300 \text{ m} - 611.73 \text{ Pa}) \\ &\approx 271 \text{ K} \end{aligned}$$

Then use Equation (21) from the script:

$$T(z) = T_s + \frac{\sqrt{\pi}}{2} l \left( \frac{dT}{dz} \right)_B \left[ \operatorname{erf} \left( \frac{z}{l} \right) - \operatorname{erf} \left( \frac{H}{l} \right) \right].$$

In order for a lake to form we need  $T(0 \text{ m}) = T_m$ .

$$\begin{aligned} T(0 \text{ m}) &= T_s + \frac{\sqrt{\pi}}{2} l \left( \frac{dT}{dz} \right)_B \left[ \operatorname{erf} \left( \frac{z}{l} \right) - \operatorname{erf} \left( \frac{H}{l} \right) \right] \\ &= T_s + \frac{\sqrt{\pi}}{2} l \left( \frac{G}{k} \right) \left[ \operatorname{erf} \left( \frac{z}{l} \right) - \operatorname{erf} \left( \frac{H}{l} \right) \right] \end{aligned}$$

The geothermal flux at the base thus must be equal or larger:

$$G \geq \frac{T_s - T(0 \text{ m})}{\frac{\sqrt{\pi}}{2} l \left( \frac{1}{k} \right) \left[ -\operatorname{erf} \left( \frac{H}{l} \right) \right]} \approx 0.5 \text{ W m}^2$$