Numerical modelling of ice sheets, streams, and shelves

Lectures at McCarthy, Alaska, August 2014

Ed Bueler
Dept of Math & Stat, and Geophysical Inst
University of Alaska, Fairbanks
elbueler@alaska.edu

what are you trying to do with this thing?

slogans & scope

slogans: I will

- focus on approximating ice flow
- provide numerical codes that actually work
- always care about the continuum model

scope: I will cover these

- models
  - shallow ice approximation (SIA) in 2D
  - shallow shelf approximation (SSA) in 1D
  - mass continuity & surface kinematical equations
- numerical ideas
  - finite difference schemes
  - solving algebraic systems from stress balances
  - verification

notation

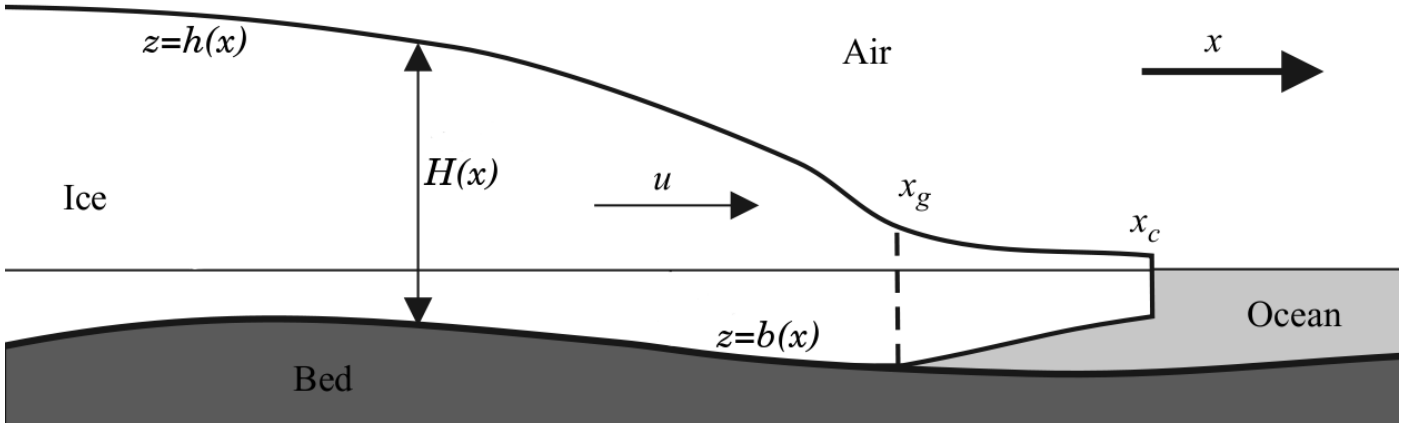


figure modified from Schoof (2007)

- coordinates t, x, y, z (with z vertical, positive upward)
- subscripts for partial derivatives $u_x = \partial u / \partial x$
- $H =$ ice thickness
- $h =$ ice surface elevation
- $b =$ bedrock surface elevation
- $T =$ temperature
- $\mathbf{u} = (u, v, w) =$ ice velocity
- $\rho =$ density of ice
- $\rho_w =$ density of ocean water
- $g =$ acceleration of gravity
- $n = 3$ Glen flow law exponent $= 3$
- $A = A(T) =$ ice softness in Glen law ($\mathbf{D}_{ij} = A(T)\tau^{n-1}\tau_{ij}$)
- please ask about notation! (stupid questions impossible)

Matlab/Octave codes

- lectures and notes are structured around 18 ice flow codes
- several codes will appear in these lectures, but not all
- each is $\sim$ 1/2 page of Matlab/Octave code
- please give them a try!
  - `.zip` and `.tar.gz` forms available from memory stick
  - and online:

`https://github.com/bueler/karthaus`

Outline

introduction: view from outside glaciology

shallow ice sheets

mass continuity

shelves and streams

ice in glaciers is a *fluid*

- what's a fluid?

- at minimum, we describe fluids by
  - a *density field* $\rho(t, x, y, z)$
  - a vector *velocity field* $\mathbf{u}(t, x, y, z)$

ice in glaciers is a *fluid* 2

- if ice fluid were
  - faster-moving than it actually is, and
  - linearly-viscous like liquid water

  then ice flow would be a "typical" fluid

- for typical fluids one uses the Navier-Stokes equations as the model:

$$\nabla \cdot \mathbf{u} = 0 \qquad \textit{incompressibility}$$

$$\rho\left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \nabla \cdot \tau_{ij} + \rho \mathbf{g} \qquad \textit{force balance}$$

$$2\nu \mathbf{D}_{ij} = \tau_{ij} \qquad \textit{flow law}$$

- force balance equation is "$ma = F$"

hmmm . . . does not sound like glaciology to me!

- yes, numerical ice sheet flow modelling is “computational fluid dynamics”
  - it’s large-scale like atmosphere and ocean
  - . . . but it is a weird one
- consider what makes atmosphere/ocean flow exciting:
  - turbulence
  - convection
  - coriolis force
  - density/salinity variation
  - chemistry (methane, ozone, . . .)
- none of the above list is relevant to ice flow
- so what could be interesting about the flow of slow, cold, stiff, laminar, inert old ice?
  - it’s *ice dynamics!*

ice is a slow, shear-thinning fluid

- our fluid is

  slow: $\rho\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) \approx 0$

  non-Newtonian: viscosity ν is not constant

- "slow":

$$\rho\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) \approx 0 \qquad \Longleftrightarrow \qquad \begin{pmatrix}\text{forces of inertia}\\ \text{are neglected}\end{pmatrix}$$

- non-Newtonian in a "shear-thinning" way
  - higher strain rates means lower viscosity

- *so* the standard "full" model is Glen-law ($n = 3$) Stokes:

$$\begin{aligned} \nabla\cdot\mathbf{u} &= 0 && \textit{incompressibility} \\ 0 &= -\nabla p + \nabla\cdot\tau_{ij} + \rho\,\mathbf{g} && \textit{force balance} \\ \mathbf{D}_{ij} &= A\tau^2\tau_{ij} && \textit{flow law} \end{aligned}$$

“slow” means no memory of velocity (i.e. momentum)

- a time-stepping ice sheet code . . .
  - recomputes the full velocity field at every time step, and
  - does not require velocity from the previous step[1]
- because there is no memory of previous velocity, *velocity is a “diagnostic” output of ice flow models*

[1] to be a weatherman you’ve got to know which way the wind blows . . . but don’t expect that from a glaciologist

plane flow Stokes

- recall the Glen-law ($n = 3$) Stokes model:

$$\nabla \cdot \mathbf{u} = 0 \qquad \textit{incompressibility}$$
$$0 = -\nabla p + \nabla \cdot \tau_{ij} + \rho\,\mathbf{g} \qquad \textit{force balance}$$
$$\mathbf{D}_{ij} = A\tau^2 \tau_{ij} \qquad \textit{flow law}$$

- now work in a x, z plane
  - like the centerline of a glacier
  - or in a cross-flow plane
- notation on next slide:
  - x, z subscripts are partial derivatives
  - τ_{13} is the "vertical" shear stress
  - τ_{11} and $\tau_{33} = -\tau_{11}$ are (deviatoric) longitudinal stresses

plane flow Stokes 2

- in the x, z plane flow case the Stokes equations say

$$
\begin{aligned}
u_x + w_z &= 0 && \textit{incompressibility} \\
p_x &= \tau_{11,x} + \tau_{13,z} && \textit{stress balance } (x) \\
p_z &= \tau_{13,x} - \tau_{11,z} - \rho g && \textit{stress balance } (z) \\
u_x &= A\tau^2 \tau_{11} && \textit{flow law } \text{(diagonal)} \\
u_z + w_x &= 2A\tau^2 \tau_{13} && \textit{flow law } \text{(off-diagonal)}
\end{aligned}
$$

- we have five equations in five unknowns $(u, w, p, \tau_{11}, \tau_{13})$
- complicated enough . . . what about in a simplified situation?

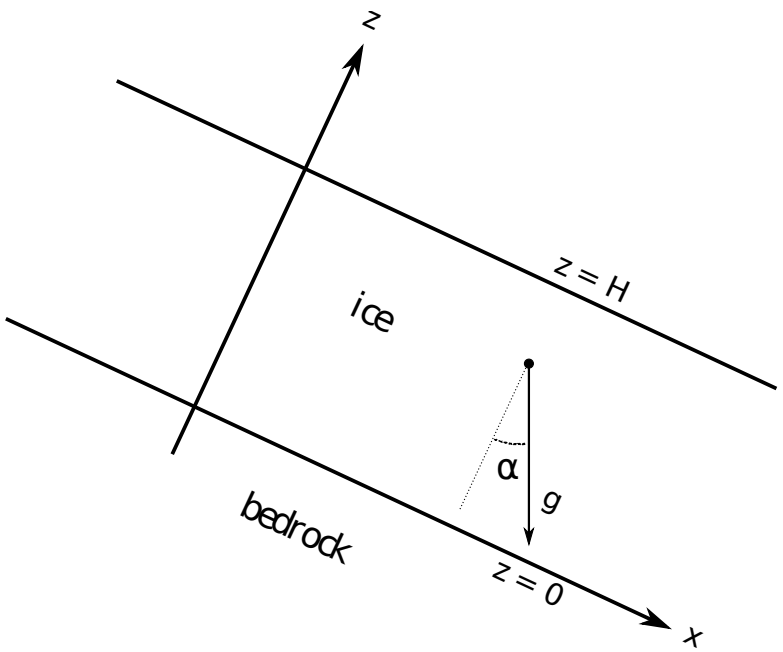
slab-on-a-slope

- suppose we have constant thickness and tilt the bed
- rotated coordinates:

$$\mathbf{g} = g \sin\theta \, \hat{x} - g \cos\theta \, \hat{z}$$

- so p_x, p_z equations are now:

$$p_x = \tau_{11,x} + \tau_{13,z} + \rho g \sin\theta$$
$$p_z = \tau_{13,x} - \tau_{11,z} - \rho g \cos\theta$$



- for such a slab-on-a-slope there is *no variation in x*
- the equations simplify:

$$\boxed{\begin{aligned} w_z &= 0 & 0 &= \tau_{11} \\ \tau_{13,z} &= -\rho g \sin\theta & u_z &= 2A\tau^2\tau_{13} \\ p_z &= -\rho g \cos\theta \end{aligned}}$$

slab-on-a-slope 2

- add some boundary conditions:

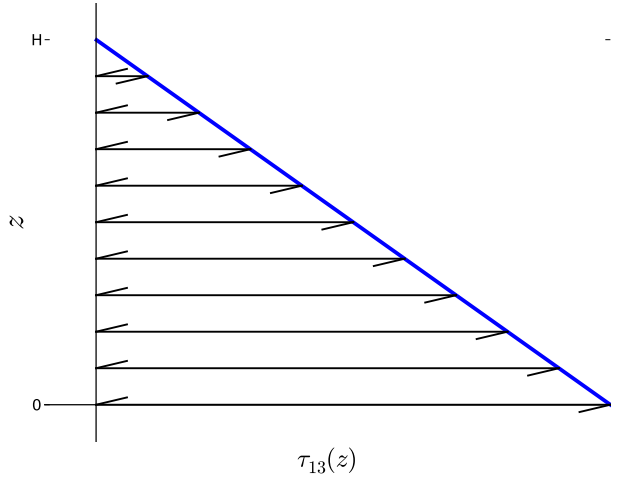
$$w(\text{base}) = 0, \qquad p(\text{surface}) = 0, \qquad u(\text{base}) = u_0$$

- by integrating vertically, get:

$$w = 0, \qquad p = \rho g \cos\theta (H - z), \qquad \tau_{13} = \rho g \sin\theta (H - z)$$

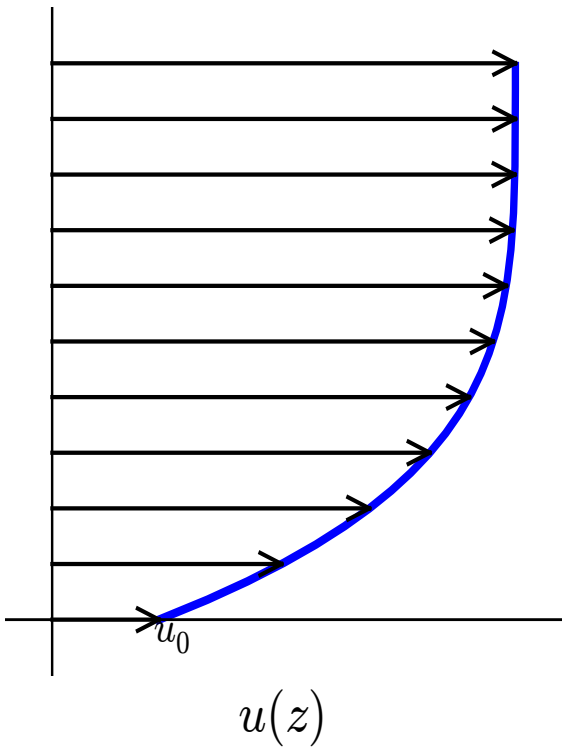
- and from "$u_z = 2A\tau^2\tau_{13}$" get velocity formula

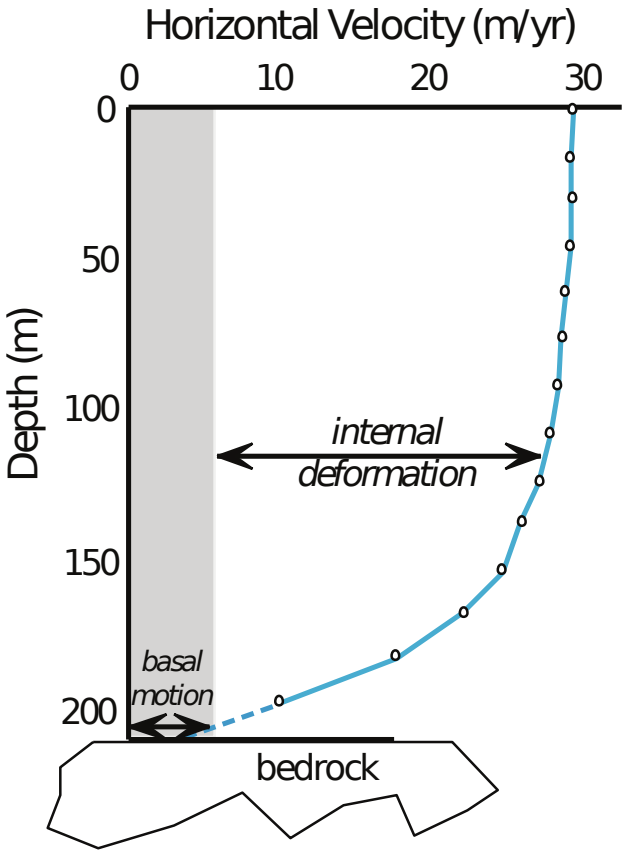
$$u(z) = u_0 + 2A(\rho g \sin\theta)^3 \int_0^z (H - z')^3 \, dz'$$
$$= u_0 + \frac{1}{2} A(\rho g \sin\theta)^3 \left(H^4 - (H - z)^4\right)$$



slab-on-a-slope 3

- do we believe these equations?
- velocity formula on last slide gives figure below
- compare to observations at right





Velocity profile of the Athabasca Glacier, Canada, derived from inclinometry (Savage and Paterson, 1963)

mass continuity

- now we know the velocity $u = u(t, x, z)$ ... so what?
- suppose, instead of slab-on-a-slope, that our ice flow has variable thickness $H(t, x)$
- compute the vertical average of velocity:

$$\bar{u}(t, x) = \frac{1}{H} \int_0^H u(t, x, z)\, dz$$

mass continuity 2

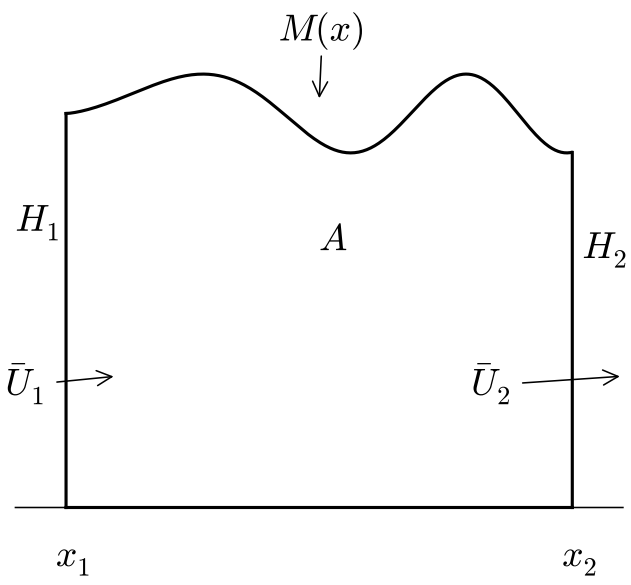
- $M(x)$ is climatic (surface) mass balance at x
- consider change of area in the figure:

$$\frac{dA}{dt} \stackrel{*}{=} \int_{x_1}^{x_2} M(x)\,dx + \bar{u}_1 H_1 - \bar{u}_2 H_2$$

- assume width $dx = x_2 - x_1$ is small so $A \approx dx\, H$
- divide eqn $*$ by dx and get

$$H_t = M - (\bar{u}H)_x$$

- this is a *mass continuity equation*



- "area" in 2D becomes "volume" in 3D

combine velocity and mass-continuity equations so far . . .

- from slab-on-slope velocity formula in $u_0 = 0$ case,

$$\bar{u}H = \int_0^H \frac{1}{2} A(\rho g \sin\theta)^3 \left(H^4 - (H-z)^4\right)\, dz$$
$$= \frac{2}{5} A(\rho g \sin\theta)^3 H^5$$

- note $\sin\theta \approx \tan\theta = -h_x$
- combine with mass continuity $H_t = M - (\bar{u}H)_x$ to get:

$$H_t = M + \left(\frac{2}{5}(\rho g)^5 A H^5 |h_x|^2 h_x\right)_x$$

- this is a rough explanation of "shallow ice approximation" (SIA) . . . next

Outline

introduction: view from outside glaciology

shallow ice sheets

mass continuity

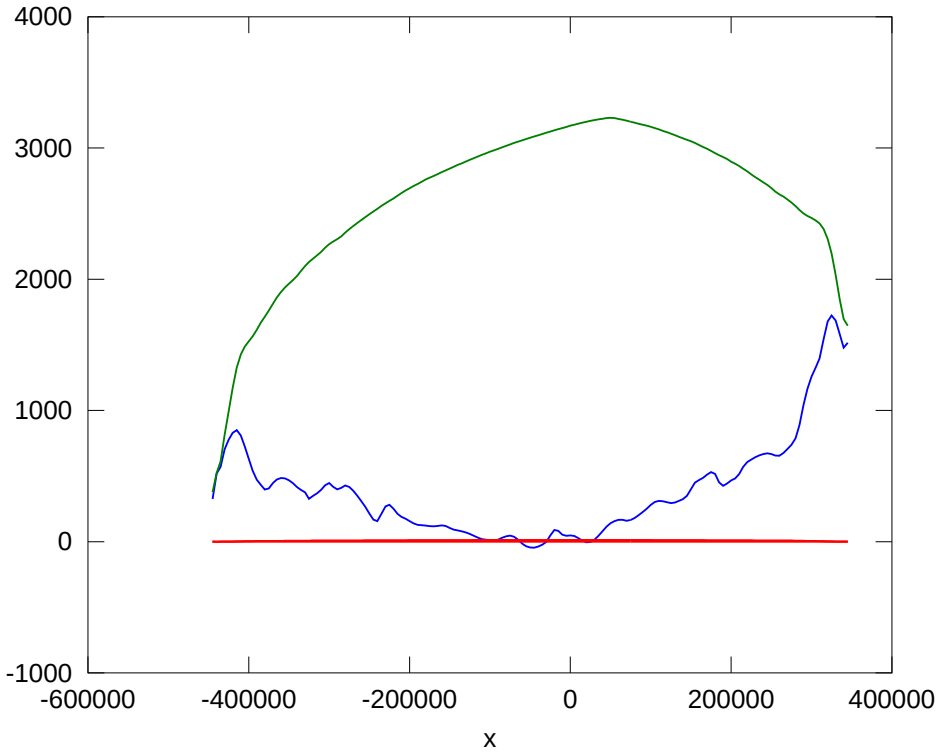
shelves and streams

slow, non-Newtonian, shallow, and sliding

- ice sheets have four outstanding properties *as fluids*:
  1. slow
  2. non-Newtonian
  3. shallow (usually)
  4. contact slip (sometimes)

regarding "shallow"

- below in red is a no-vertical-exaggeration cross section of Greenland at 71°
- green and blue: standard vertically-exaggerated cross section
- you can scale Stokes equation using smallness of $\epsilon = [H]/[L]$, where $[H]$ is a typical thickness of an ice sheet and $[L]$ is a typical horizontal dimension, . . . (Fowler, 1997)



flow model I: non-sliding, isothermal shallow ice approximation = (SIA)

a model which applies to

- small depth-to-width ratio ("shallow") grounded ice sheets
- on not-too-rough bed topography,
- whose flow is not dominated by sliding and/or liquid water at the base or margin

"Polaris Glacier," northwest Greenland, photo 122, Post & LaChapelle (2000)

SIA model equations

- though the best explanation of the SIA is to use shallowness to simplify the Stokes equations, here we take the simple slogan:

  the SIA uses the formulas from slab-on-a-slope

- shear stress approximation:

$$(\tau_{13}, \tau_{23}) = -\rho g (h - z) \nabla h$$

- let $\mathbf{u} = (u, v)$, the horizontal velocity
- we further approximate

$$\begin{aligned} \mathbf{u}_z &= 2A |(\tau_{13}, \tau_{23})|^{n-1} (\tau_{13}, \tau_{23}) \\ &= -2A (\rho g)^n (h - z)^n |\nabla h|^{n-1} \nabla h \end{aligned}$$

- by integrating vertically, in the non-sliding case,

$$\mathbf{u} = -\frac{2A(\rho g)^n}{n+1} \left[H^{n+1} - (h - z)^{n+1} \right] |\nabla h|^{n-1} \nabla h$$

- but mass continuity remains, $H_t = M - (\overline{\mathbf{u}} H)_x$

SIA thickness equation

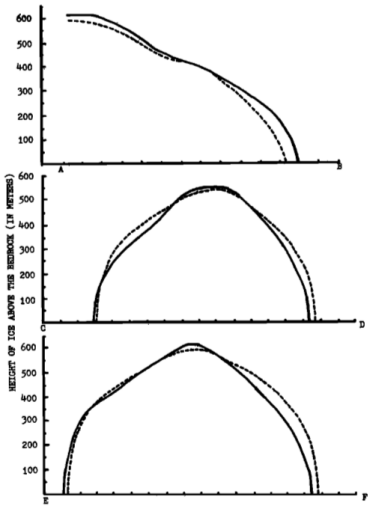
- from last slide, we get the non-sliding, isothermal shallow ice approximation for how thickness changes:

$$\boxed{H_t = M + \nabla \cdot \left(\Gamma H^{n+2} |\nabla h|^{n-1} \nabla h\right)} \qquad (1)$$

  - where H is ice thickness, h is ice surface elevation, b is bed elevation ($h = H + b$)
  - M combines surface and basal mass (im)balance: accumulation if $M > 0$, ablation if $M < 0$
  - n is the exponent in the Glen flow law
  - $\Gamma = 2A(\rho g)^n/(n+2)$ is a positive constant
- numerically solve (1) and you've got a usable model for ... *the Barnes ice cap* (Mahaffy, 1976)

good questions:

1. where does equation (1) come from?
2. how to solve it numerically?
3. how to *think* about it?



heat equation

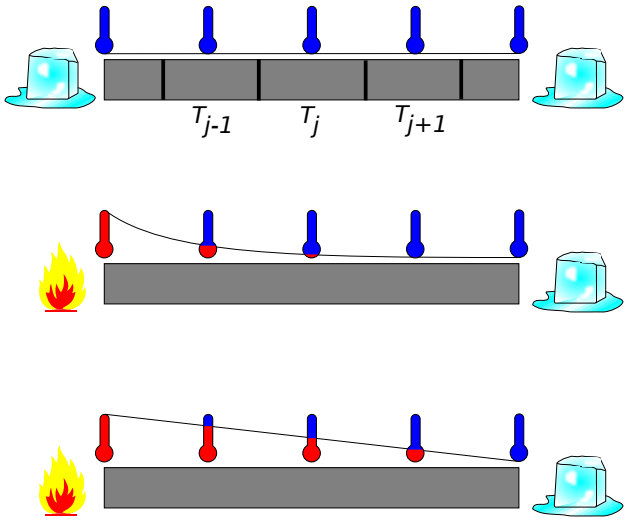
- for understanding SIA, recall heat equation
- recall Newton's law of cooling

$$\frac{dT}{dt} = -K(T - T_{\text{ambient}})$$

where T is object temperature and K relates to material and geometry of object (e.g. cup of coffee)

- Newton's law for segments of a rod:

$$\frac{dT_j}{dt} = -K\left(T_j - \frac{1}{2}(T_{j-1} + T_{j+1})\right)$$
$$= \frac{K}{2}(T_{j-1} - 2T_j + T_{j+1})$$



- this has limit as segments shrink:

$$T_t = DT_{xx}$$

- compare: finite difference approximations to derivatives

analogy: SIA versus 2D heat equation

- side-by-side comparison:

SIA: $H(t,x,y)$ is ice thickness

$$\boxed{H_t = M + \nabla \cdot \left(\Gamma H^{n+2} |\nabla h|^{n-1} \nabla h \right)}$$

heat: $T(t,x,y)$ is temperature

$$\boxed{T_t = F + \nabla \cdot (D \nabla T)}$$

- we identify the diffusivity in the SIA:

$$D = \Gamma H^{n+2} |\nabla h|^{n-1}$$

- *non-sliding shallow ice flow diffuses the ice sheet*
- some issues with this analogy:
  - D depends on solution $H(t,x,y)$
  - $D \to 0$ at margin, where $H \to 0$
  - $D \to 0$ at divides/domes, where $|\nabla h| \to 0$

numerics for heat equation: basic ideas of finite differences

- numerical schemes for heat equation are good start for SIA
- for differentiable $f(x)$ and any h, *Taylor's theorem* says

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{3!}f'''(x)h^3 + \dots$$

- you can replace "h" by multiples of Δx, e.g.:

$$f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2 - \frac{1}{3!}f'''(x)\Delta x^3 + \dots$$

$$f(x+2\Delta x) = f(x) + 2f'(x)\Delta x + 2f''(x)\Delta x^2 + \frac{4}{3}f'''(x)\Delta x^3 + \dots$$

- *combine expressions like these to give approximations of derivatives, from values on a grid*

finite differences for partial derivatives

- we want partial derivative expressions, for example with any function $u = u(t, x)$:

$$u_t(t,x) = \frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} + O(\Delta t),$$
$$u_t(t,x) = \frac{u(t+\Delta t, x) - u(t-\Delta t, x)}{2\Delta t} + O(\Delta t^2),$$
$$u_x(t,x) = \frac{u(t, x+\Delta x) - u(t, x-\Delta x)}{2\Delta x} + O(\Delta x^2),$$
$$u_{xx}(t,x) = \frac{u(t, x+\Delta x) - 2u(t,x) + u(t, x-\Delta x)}{\Delta x^2} + O(\Delta x^2)$$

  and so on
- sometimes we want a derivative in-between grid points:

$$u_x(t, x + (\Delta x/2)) = \frac{u(t, x+\Delta x) - u(t,x)}{\Delta x} + O(\Delta x^2)$$

- "$+O(h^2)$" is better than "$+O(h)$" if h is a small number

explicit scheme for heat equation

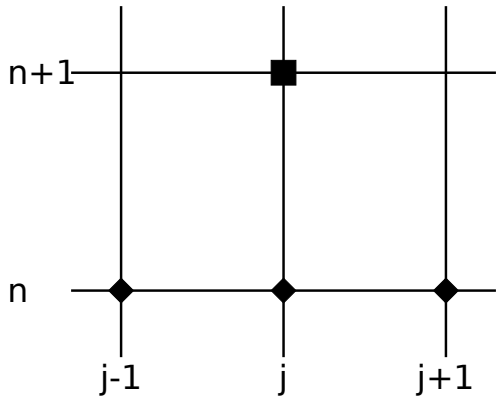
- consider 1D heat equation $T_t = DT_{xx}$
- an *explicit* scheme comes from:

$$\frac{T(t+\Delta t, x) - T(t,x)}{\Delta t} \approx D\,\frac{T(t, x+\Delta x) - 2T(t,x) + T(t, x-\Delta x)}{\Delta x^2}$$

- the difference between the equation $T_t = DT_{xx}$ and the scheme is $O(\Delta t, \Delta x^2)$ (Morton and Mayers, 2005)
- notation: (t_n, x_j) is a point in the time-space grid
- notation: $T_j^n \approx T(t_n, x_j)$
- let $\nu = D\Delta t/(\Delta x)^2$, so scheme is

$$T_j^{n+1} = \nu T_{j+1}^n + (1-2\nu)T_j^n + \nu T_{j-1}^n$$

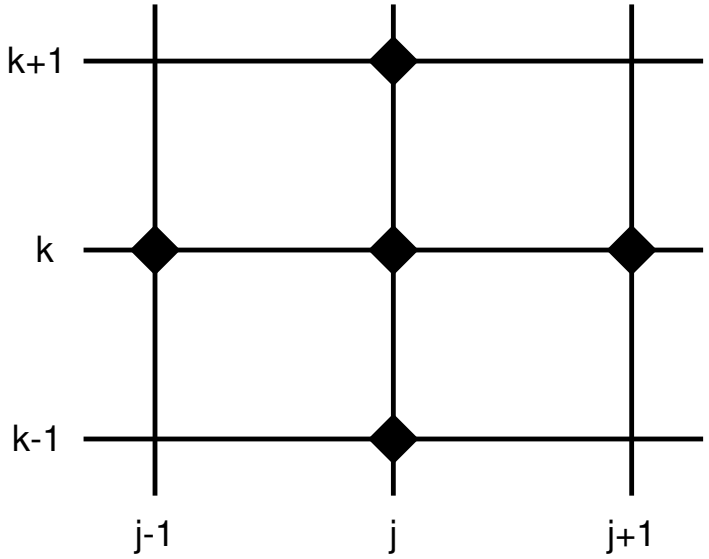
- scheme has stencil at right $\rightarrow$



explicit scheme in two space dimensions

- recall heat equation in 2D: $T_t = D(T_{xx} + T_{yy})$
- in two spatial variables we write $T^n_{jk} \approx T(t_n, x_j, y_k)$
- so the 2D explicit scheme is

$$\frac{T^{n+1}_{jk} - T^n_{jk}}{\Delta t} = D\left(\frac{T^n_{j+1,k} - 2T^n_{jk} + T^n_{j-1,k}}{\Delta x^2} + \frac{T^n_{j,k+1} - 2T^n_{jk} + T^n_{j,k-1}}{\Delta y^2}\right)$$



implementation

```
function T = heat(D,J,K,dt,N)

dx = 2 / J;     dy = 2 / K;
[x,y] = meshgrid(-1:dx:1, -1:dy:1);
T = exp(-30*(x.*x + y.*y));

mu_x = dt * D / (dx*dx);
mu_y = dt * D / (dy*dy);
for n=1:N
   T(2:J,2:K) = T(2:J,2:K) + ...
       mu_x * ( T(3:J+1,2:K) - 2 * T(2:J,2:K) + T(1:J-1,2:K) ) + ...
       mu_y * ( T(2:J,3:K+1) - 2 * T(2:J,2:K) + T(2:J,1:K-1) );
end

surf(x,y,T),  shading('interp'),  xlabel x,  ylabel y
```

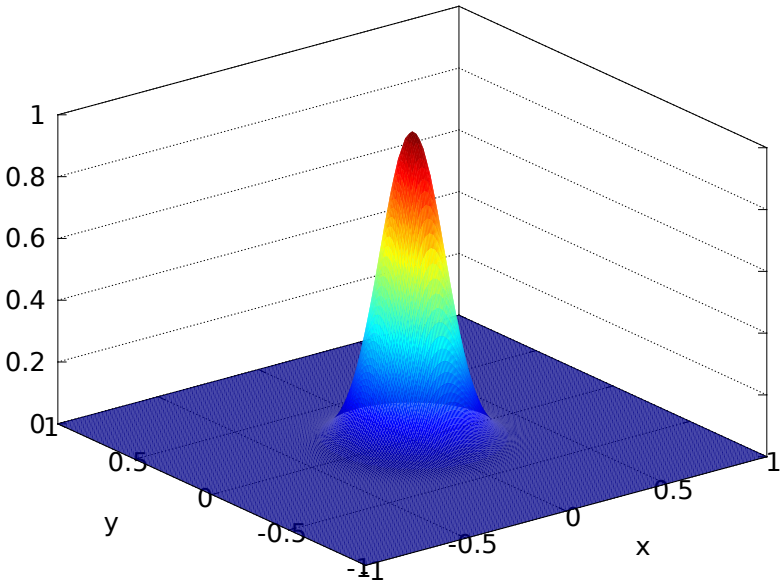
heat.m

- solves $T_t = D(T_{xx} + T_{yy})$ on square $-1 < x < 1$, $-1 < y < 1$
- example uses initial condition $T_0(x, y) = e^{-30r^2}$
- code uses "colon notation" to remove loops (over space)
- » `heat(1.0,30,30,0.001,20)`
  approximates T on 30×30 spatial grid, with $D = 1$ and $N = 20$ steps of $\Delta t = 0.001$

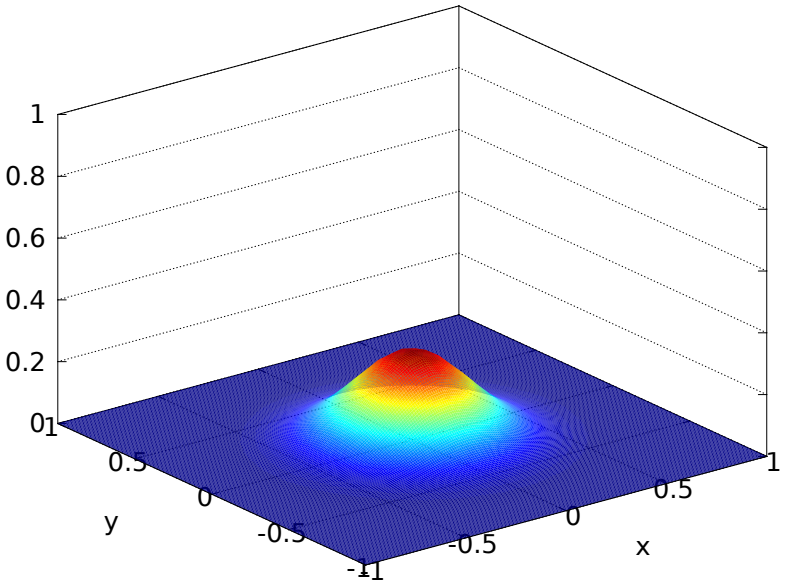
the look of success

- solving $T_t = D(T_{xx} + T_{yy})$ on 30×30 grid

initial condition $T(0, x, y)$

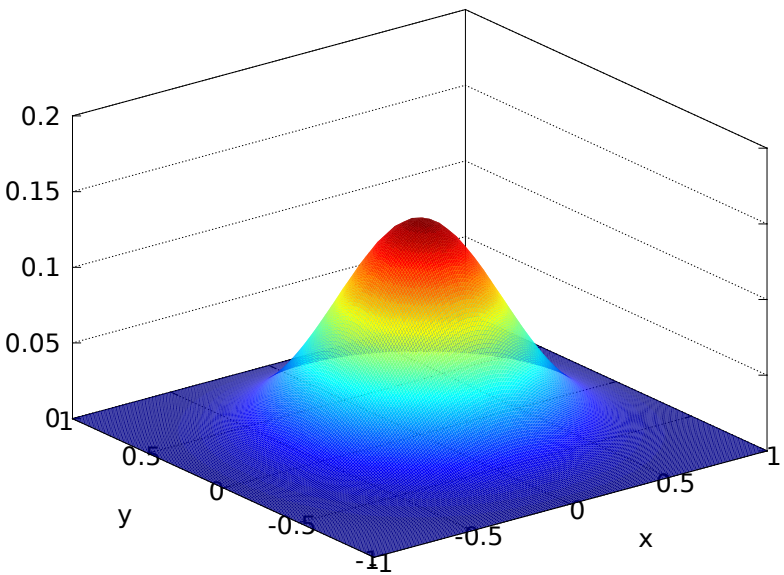


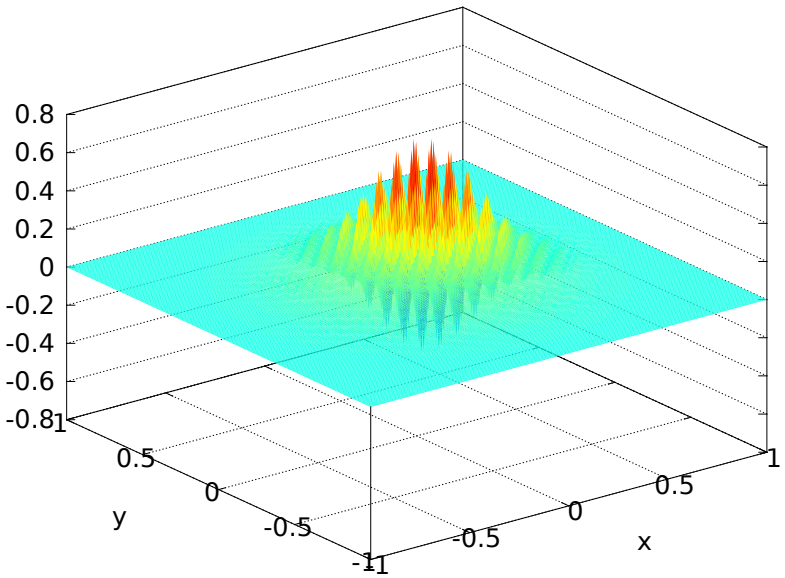
approximate solution $T(t, x, y)$
at $t = 0.02$ with $\Delta t = 0.001$



the look of instability

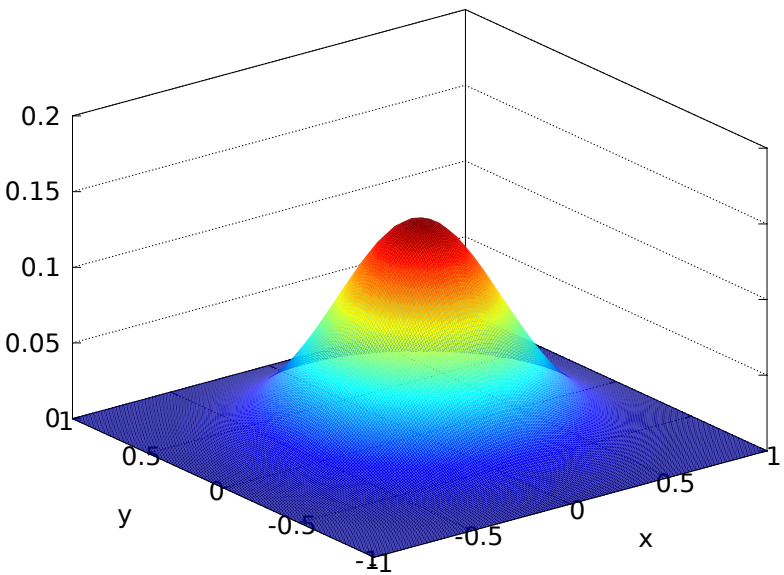
- both figures are from solving $T_t = D(T_{xx} + T_{yy})$ on the same space grid, but with slightly different time steps



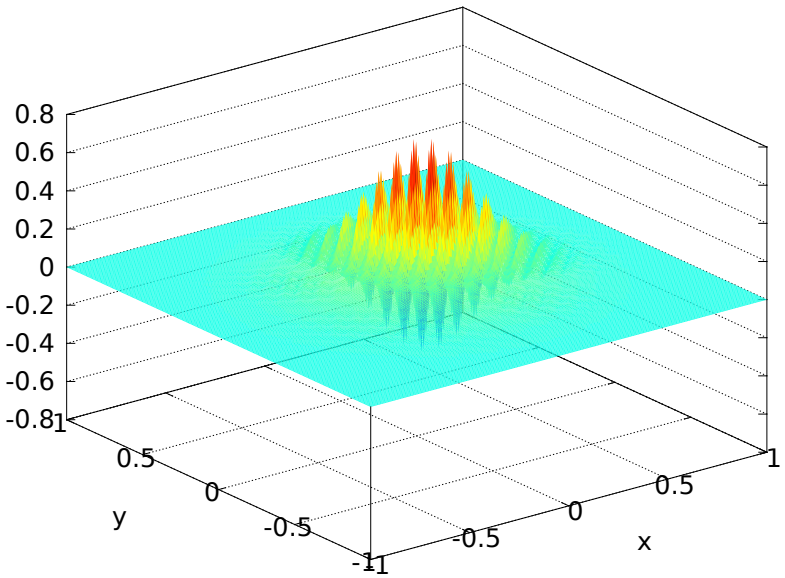


the look of instability

- both figures are from solving $T_t = D(T_{xx} + T_{yy})$ on the same space grid, but with slightly different time steps



$$\frac{D\Delta t}{\Delta x^2} = 0.402$$



$$\frac{D\Delta t}{\Delta x^2} = 0.625$$

avoid the instability

- recall 1D explicit scheme had the form

$$T_j^{n+1} = \nu T_{j+1}^n + (1-2\nu) T_j^n + \nu T_{j-1}^n$$

- thus the new value u_j^{n+1} is an *average* of the old values, *if the middle coefficient is positive*:

$$1 - 2\nu \geq 0 \quad \Longleftrightarrow \quad \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2} \quad \Longleftrightarrow \quad \Delta t \leq \frac{\Delta x^2}{2D}$$

- averaging is always stable because averaged wiggles are always smaller than the original wiggles
- ...so this condition is a sufficient *stability criterion*
- so:

  the result was unstable because the time step was too big

adaptive implementation: guaranteed stability

```
function T = heatadapt(D,J,K,tf)

dx = 2 / J;     dy = 2 / K;
[x,y] = ndgrid(-1:dx:1, -1:dy:1);
T = exp(-30*(x.*x + y.*y));

t = 0.0;     count = 0;
while t < tf
   dt0 = 0.25 * min(dx,dy)^2 / D;
   dt = min(dt0, tf - t);
   mu_x = dt * D / (dx*dx);     mu_y = dt * D / (dy*dy);
   T(2:J,2:K) = T(2:J,2:K) + ...
       mu_x * ( T(3:J+1,2:K) - 2 * T(2:J,2:K) + T(1:J-1,2:K) ) + ...
       mu_y * ( T(2:J,3:K+1) - 2 * T(2:J,2:K) + T(2:J,1:K-1) );
   t = t + dt;
   count = count + 1;
end

surf(x,y,T),  shading('interp'),  xlabel x,  ylabel y
```

`heatadapt.m`

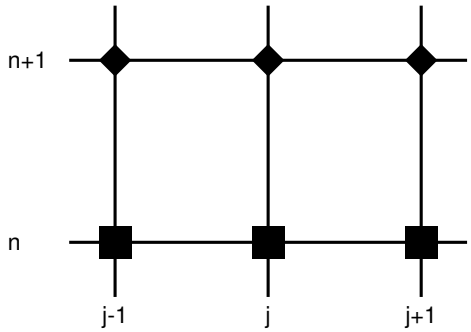
- same as `heat.m` except

  choose time step from stability criterion

alternative instability fix: implicitness

- implicit methods can be stable for *any* positive time step Δt

- an implicit scheme is *Crank-Nicolson* $\longrightarrow$
- Crank-Nicolson has smaller error too: $O(\Delta t^2, \Delta x^2)$



- *but* you have to solve linear (or nonlinear) systems of equations to take each time step

- Donald Knuth has advice for ice sheet modelers:

  We should forget about small efficiencies . . . : premature optimization is the root of all evil.

variable diffusivity and time steps

- recall the analogy: (SIA) $\leftrightarrow$ (heat eqn)
- the SIA has a diffusivity which varies in space, so consider a more general heat equation:

$$T_t = F + \nabla \cdot (D(x,y)\nabla T)$$

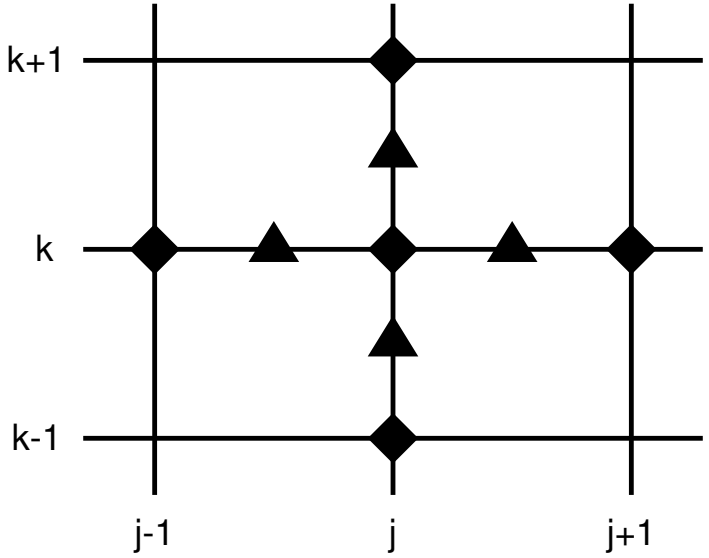
- the explicit method is conditionally stable with the same time step restriction if we evaluate diffusivity $D(x,y)$ at staggered grid points:

$$\nabla \cdot (D(x,y)\nabla u) \approx \frac{D_{j+1/2,k}(T_{j+1,k} - T_{j,k}) - D_{j-1/2,k}(T_{j,k} - T_{j-1,k})}{\Delta x^2} + \frac{D_{j,k+1/2}(T_{j,k+1} - T_{j,k}) - D_{j,k-1/2}(T_{j,k} - T_{j,k-1})}{\Delta y^2}$$

in stencil at right:

diamonds: T

triangles: D



general diffusion equation code

```
function [T,dtav] = diffusion(Lx,Ly,J,K,Dup,Ddown,Dright,Dleft,T0,tf,F,b)

dx = 2 * Lx / J;    dy = 2 * Ly / K;
[x,y] = ndgrid(-Lx:dx:Lx, -Ly:dy:Ly);
T = T0;
if nargin < 11, F = zeros(size(T0)); end
if nargin < 12, b = zeros(size(T0)); end

t = 0.0;    count = 0;
while t < tf
   maxD = [max(max(Dup)) max(max(Ddown)) max(max(Dleft)) max(max(Dright))];
   maxD = max(maxD);
   if maxD <= 0.0
     dt = tf - t;
   else
     dt0 = 0.25 * min(dx,dy)^2 / maxD;
     dt = min(dt0, tf - t);
   end
   mu_x = dt / (dx*dx);    mu_y = dt / (dy*dy);
   Tb = T + b;
   T(2:J,2:K) = T(2:J,2:K) + ...
       mu_y * Dup    .* ( Tb(2:J,3:K+1) - Tb(2:J,2:K)   ) - ...
       mu_y * Ddown  .* ( Tb(2:J,2:K)   - Tb(2:J,1:K-1) ) + ...
       mu_x * Dright .* ( Tb(3:J+1,2:K) - Tb(2:J,2:K)   ) - ...
       mu_x * Dleft  .* ( Tb(2:J,2:K)   - Tb(1:J-1,2:K) );
   T = T + F * dt;
   t = t + dt;    count = count + 1;
end
dtav = tf / count;
```

diffusion.m

- solves abstract diffusion equation $T_t = \nabla \cdot (D(x,y)\,\nabla T)$
- user supplies diffusivity on staggered grid

verification of numerical ice flow codes

- how do we make sure an *implemented* numerical scheme is correct?
  - *technique* 1: don't make any mistakes
  - *technique* 2: compare your model with others, and hope that the outliers are the ones with errors
  - *technique* 3: build-in a comparison to an exact solution, and actually measure the numerical error = verification

- where to get exact solutions for ice flow models?
  - textbook: Greve and Blatter (2009)
  - similarity solutions to SIA (Halfar 1983; Bueler et al 2005)
  - manufactured solutions to thermo-coupled SIA (Bueler et al 2007)
  - flowline and cross-flow SSA solutions (Bodvardsson, 1955; van der Veen, 1985; Schoof, 2006)
  - flowline Blatter solutions (Glowinski and Rappaz 2003)
  - flowline Stokes solutions for constant viscosity (Ladyzhenskaya 1963, Balise and Raymond 1985)
  - manufactured solutions to the Stokes equations (Sargent and Fastook 2010; Jouvet and Rappaz 2011)

exact solution of heat equation

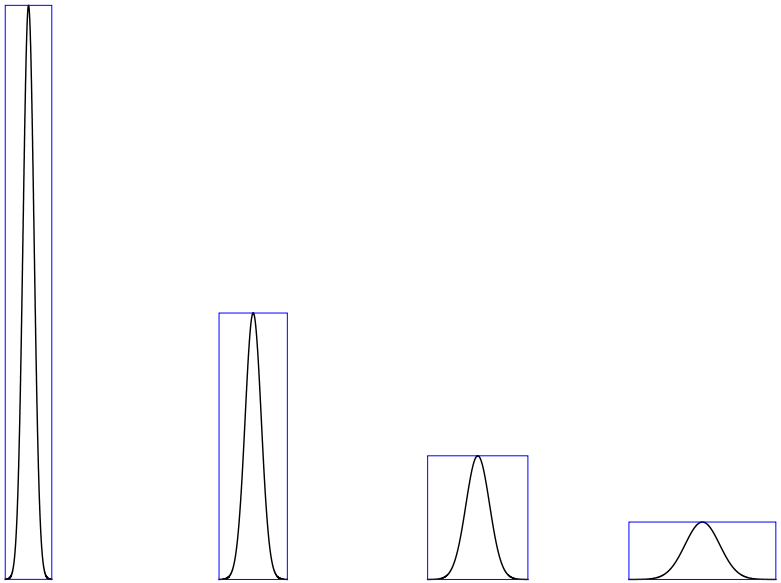
- the simple heat equation in 1D with constant diffusivity $D > 0$ is:

$$T_t = DT_{xx}$$

- many *exact* solutions to the heat equation are known
- I'll show the "Green's function" (a.k.a. "fundamental solution" or "heat kernel")
- it starts at time $t = 0$ with a "delta function" of heat at the origin $x = 0$ and then it spreads out over time
- we find it by a method which generalizes to the SIA

Green's function of heat equation

- the solution is "self-similar" over time
- as time goes it changes shape by
  - shrinking the output (vertical) axis and
  - lengthening the input (horizontal) axis
- ...but otherwise it is the same shape
- the integral over x is independent of time



increasing time $\longrightarrow$

similarity solutions

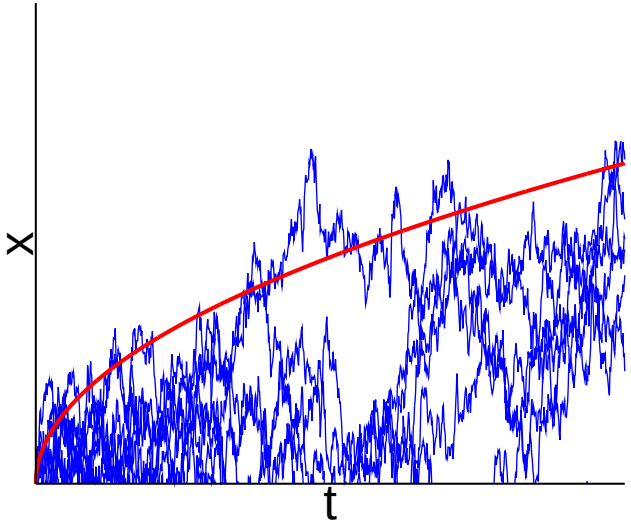
- Green's function of heat equation in 1D is

$$T(t,x) = C\,t^{-1/2} e^{-x^2/(4Dt)}$$

- "similarity" variables for 1D heat equation are

$$\underbrace{s = t^{-1/2}x}_{\textit{input scaling}}, \qquad \underbrace{T(t,x) = t^{-1/2}\phi(s)}_{\textit{output scaling}}$$

- *historical note*: in 1905 Einstein saw that the average distance traveled by particles in thermal motion scales like $\sqrt{t}$, so $s = t^{-1/2}x$ is an invariant



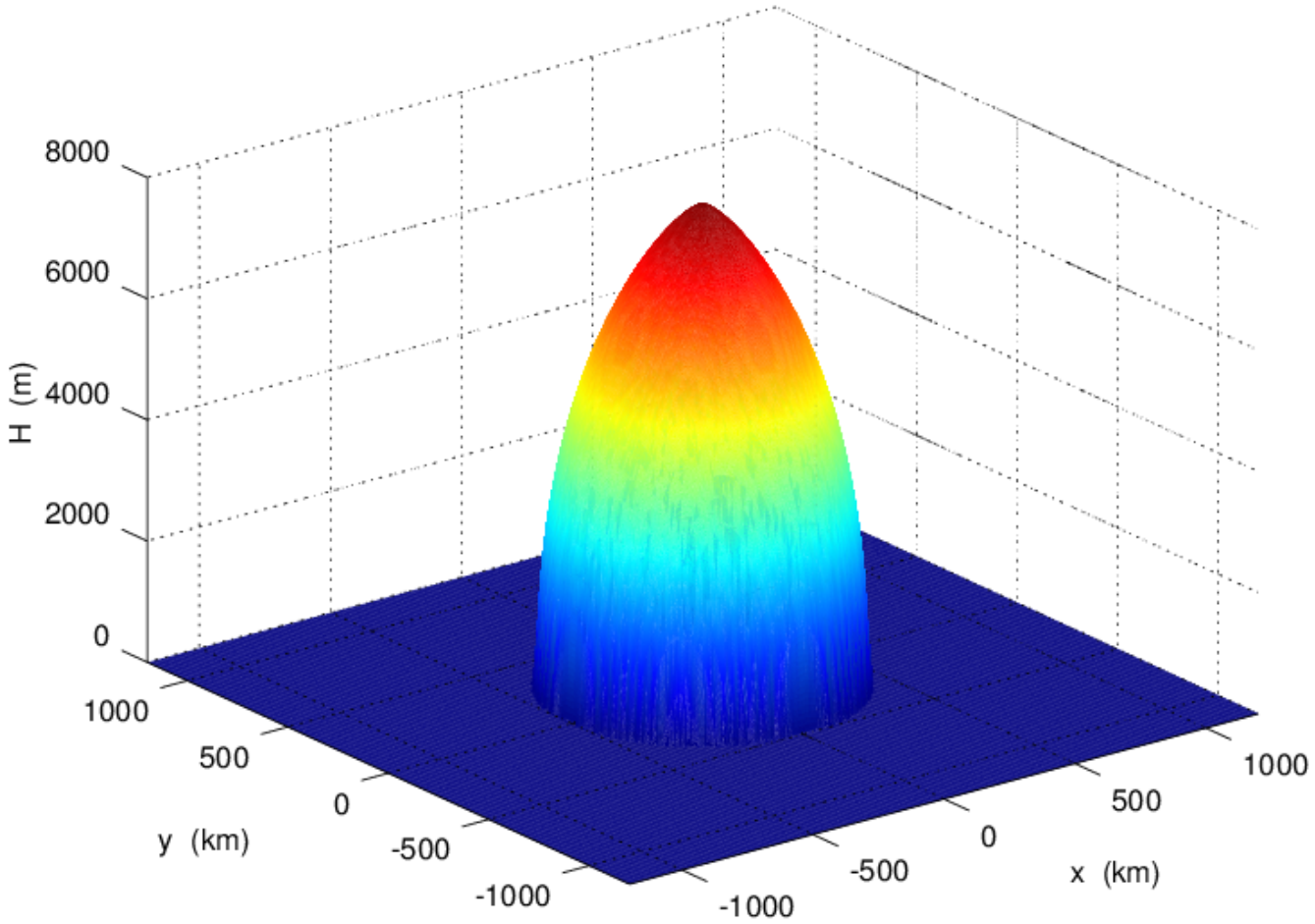
similarity solution to SIA

- jump forward to 1981
- P. Halfar found the similarity solution of the SIA in the case of flat bed and no surface mass balance
- Halfar's 2D solution for Glen flow law with $n = 3$ has scalings

$$H(t, r) = t^{-1/9}\phi(s), \qquad s = t^{-1/18}r$$

- ...so the diffusion of ice really slows down as the shape flattens out!

Halfar solution to the SIA: the movie



frames from $t = 4$ months to $t = 10^6$ years, equal spaced in *exponential* time

Halfar solution to the SIA: the formula

- for $n = 3$ the solution formula is:

$$H(t,r) = H_0 \left(\frac{t_0}{t}\right)^{1/9} \left[1 - \left(\left(\frac{t_0}{t}\right)^{1/18} \frac{r}{R_0}\right)^{4/3}\right]^{3/7}$$

- the "characteristic time" is

$$t_0 = \frac{1}{18\Gamma}\left(\frac{7}{4}\right)^3 \frac{R_0^4}{H_0^7}$$

  if H_0, R_0 are central height and ice cap radius at $t = t_0$
- you choose H_0 and R_0 and then determine t_0
- it is a simple formula to use for verification!

is the Halfar solution *good for any modeling*?

- John Nye and others (2000) compared different flow laws for the South Polar Cap on Mars
- they evaluated CO_2 ice and H_2O ice softness parameters by comparing the long-time behavior of the corresponding Halfar solutions
- conclusions:

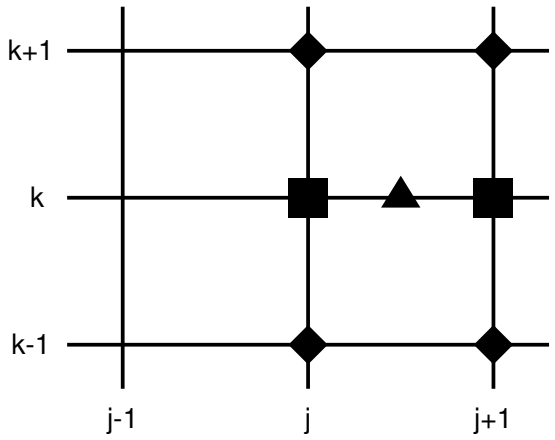
  > *. . . none of the three possible [CO_2] flow laws will allow a 3000-m cap, the thickness suggested by stereogrammetry, to survive for* 10^7 *years, indicating that the south polar ice cap is probably not composed of pure CO_2 ice . . . the south polar cap probably consists of water ice, with an unknown admixture of dust.*

computing diffusivity in SIA

- for numerical stability we compute $D = \Gamma H^{n+2} |\nabla h|^{n-1}$ on the staggered grid
- various schemes proposed (Mahaffy, 1976; van der Veen 1999; Hindmarsh and Payne 1996)
- all schemes involve
  - averaging H
  - differencing h
  - in a "balanced" way, for better accuracy,

  to get the diffusivity on staggered grid

- Mahaffy stencil $\rightarrow$



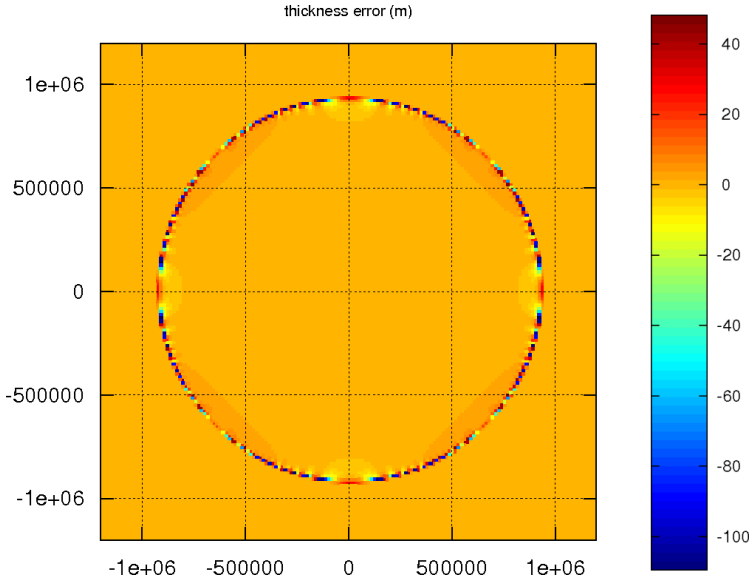
SIA implementation: flat bed case

```
function [H,dtlist] = siaflat(Lx,Ly,J,K,H0,deltat,tf)

g = 9.81;    rho = 910.0;    secpera = 31556926;
A = 1.0e-16/secpera;    Gamma  = 2 * A * (rho * g)^3 / 5;
H = H0;

dx = 2 * Lx / J;   dy = 2 * Ly / K;
N = ceil(tf / deltat);    deltat = tf / N;
j  = 2:J;    k = 2:K;
nk = 3:K+1;    sk = 1:K-1;    ej = 3:J+1;    wj = 1:J-1;

t = 0;   dtlist = [];
for n=1:N
  Hup = 0.5 * ( H(j,nk) + H(j,k) );   Hdn = 0.5 * ( H(j,k) + H(j,sk) );
  Hrt = 0.5 * ( H(ej,k) + H(j,k) );   Hlt = 0.5 * ( H(j,k) + H(wj,k) );
  a2up = (H(ej,nk) + H(ej,k) - H(wj,nk) - H(wj,k)).^2 / (4*dx)^2 + ...
         (H(j,nk) - H(j,k)).^2 / dy^2;
  a2dn = (H(ej,k) + H(ej,sk) - H(wj,k) - H(wj,sk)).^2 / (4*dx)^2 + ...
         (H(j,k) - H(j,sk)).^2 / dy^2;
  a2rt = (H(ej,k) - H(j,k)).^2 / dx^2 + ...
         (H(ej,nk) + H(j,nk) - H(ej,sk) - H(j,sk)).^2 / (4*dy)^2;
  a2lt = (H(j,k) - H(wj,k)).^2 / dx^2 + ...
         (H(wj,nk) + H(j,nk) - H(wj,sk) - H(j,sk)).^2 / (4*dy)^2;
  Dup = Gamma * Hup.^5 .* a2up;  Ddn = Gamma * Hdn.^5 .* a2dn;
  Drt = Gamma * Hrt.^5 .* a2rt;  Dlt = Gamma * Hlt.^5 .* a2lt;
  [H,dtadapt] = diffusion(Lx,Ly,J,K,Dup,Ddn,Drt,Dlt,H,deltat);
  t = t + deltat;    dtlist = [dtlist dtadapt];
end
```

siaflat.m

verifying SIA code vs Halfar

```
octave:40> verifysia(20)
average abs error              = 22.310
maximum abs error              = 227.849
octave:41> verifysia(40)
average abs error              = 9.490
maximum abs error              = 241.470
octave:42> verifysia(80)
average abs error              = 2.800
maximum abs error              = 155.796
octave:43> verifysia(160)
average abs error              = 1.059
maximum abs error              = 109.466
```



Trust but verify.

(Ronald Reagan)

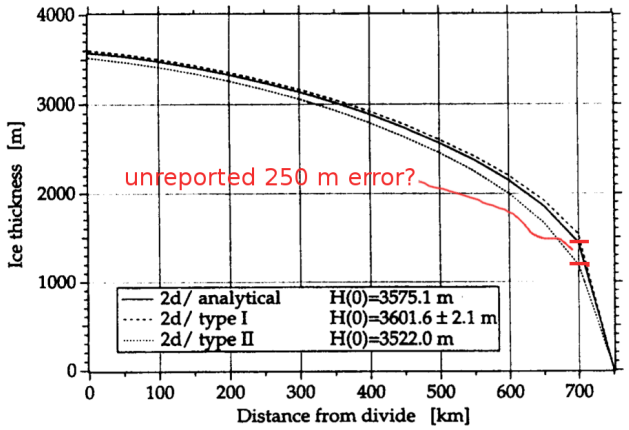
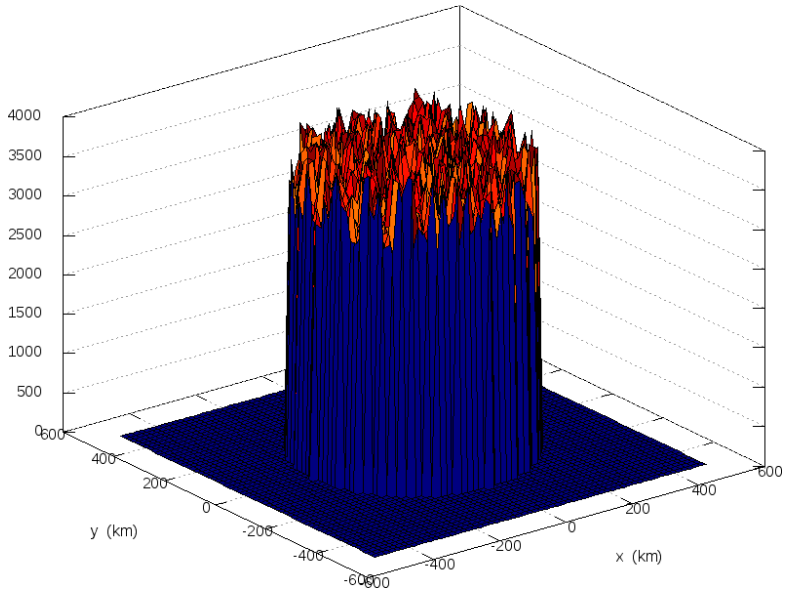
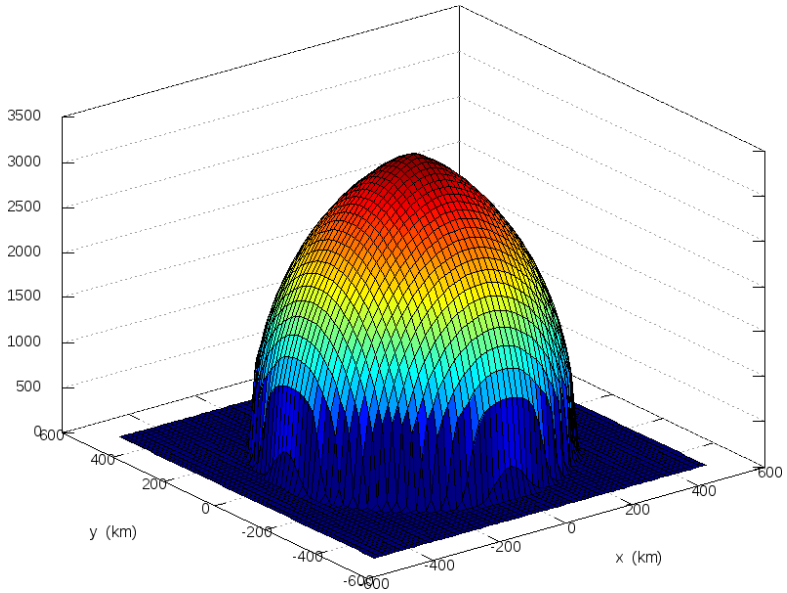


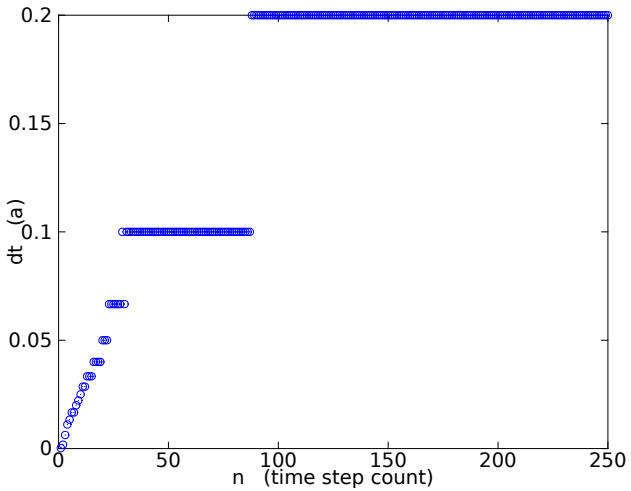
figure 2 in Huybrechts et al. (1996)

demonstrate robustness

see `roughice.m`, which calls `siaflat.m` after setting-up the nasty initial state at left:



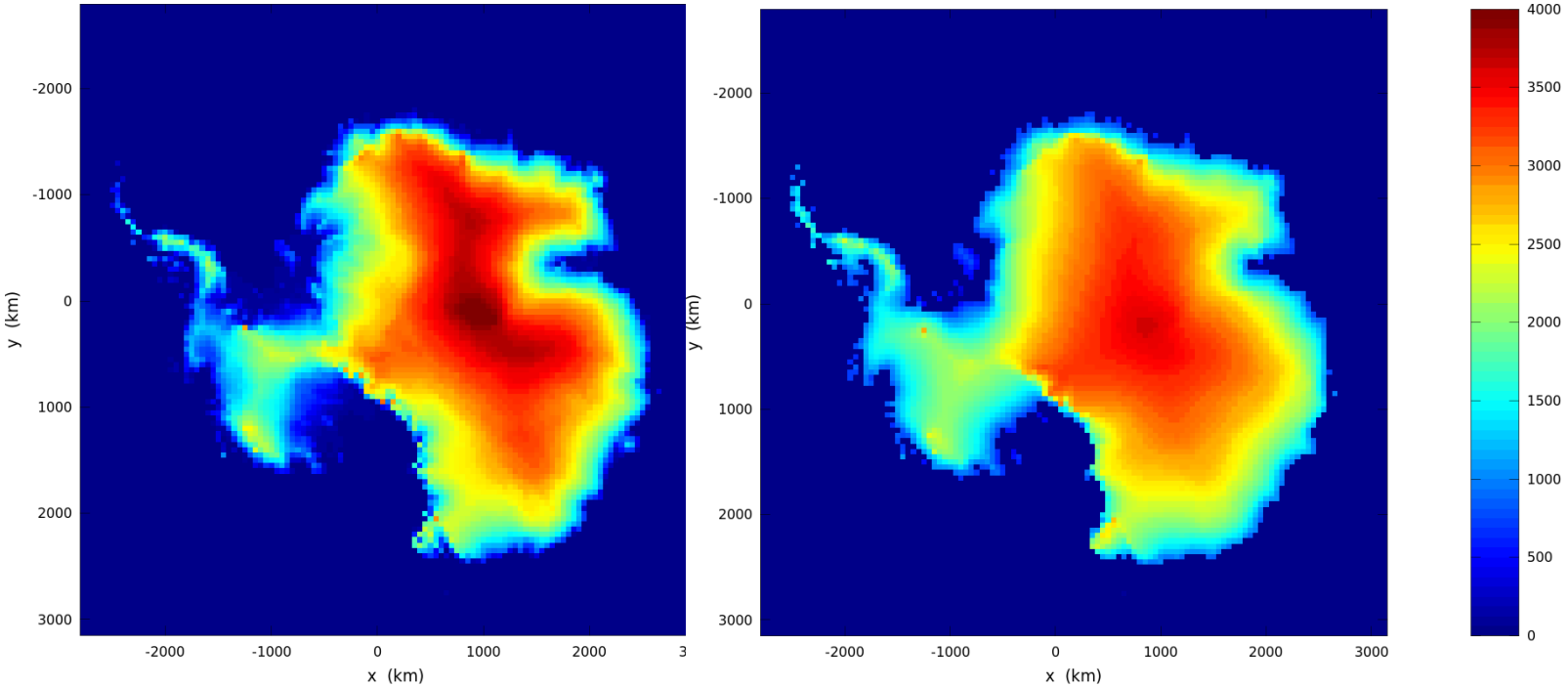




model the Antarctic ice sheet

- with careful-but-small modifications of `siaflat.m`, which make a good exercise:
  - observed accumulation as surface mass balance,
  - allow non-flat bed (so $H \neq h$),
  - compute surface slopes correctly where floating, and
  - calve at current calving front location

  here are results from this *toy* Antarctic flow model
- a 2000 model year run on a $\Delta x = 50$ km grid; runtime a few seconds



Outline

introduction: view from outside glaciology

shallow ice sheets

mass continuity

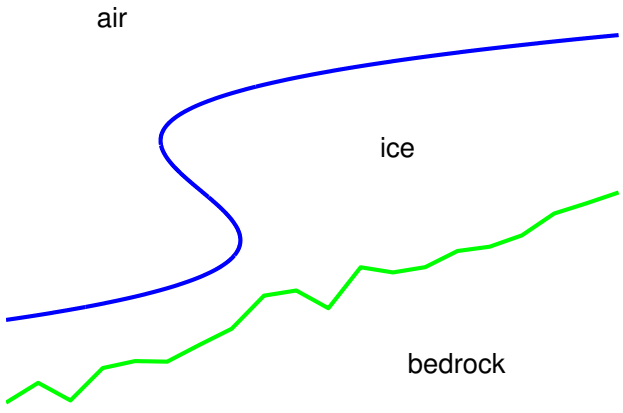
shelves and streams

the most basic shallow assumption

- there are many shallow theories: SIA, SSA, hybrids, Blatter, . . .
- *all* make one assumption not required in Stokes:

  the surface and base of the ice are given by functions $z = h(t, x, y)$ and $z = b(t, x, y)$

- surface overhang is not allowed
- most Stokes models make this assumption too?



not shallow!

not shallow!

three equations for geometry change

- let a be the climatic (surface) mass balance function; $a > 0$ is accumulation
- s be the basal melt rate function; $s > 0$ is basal melting
- let $M = a - s$: "climatic-basal mass balance function" in glossary
- define the map-plane flux of ice,

$$\mathbf{q} = \int_b^h (u, v)\, dz = \overline{\mathbf{U}}\, H$$

- the three equations for geometry change:

surface kinematical	$h_t = a - u\big\vert_h h_x - v\big\vert_h h_y + w\big\vert_h$
base kinematical	$b_t = s - u\big\vert_b b_x - v\big\vert_b b_y + w\big\vert_b$
mass continuity	$H_t = M - \nabla \cdot \mathbf{q}$

kinematic and mass continuity equations

- what does the "most basic shallow assumption" get you?
- *answer 1*: a (map-plane) mass continuity equation from the kinematical equations and incompressibility
- *answer 2*: of these three equations,
  - surface kinematical
  - base kinematical
  - mass continuity

  any two imply the third

- to show the above, recall:
  - the incompressibility of ice

$$u_x + v_y + w_z = 0$$

  - and the Leibniz rule for differentiating integrals

$$\frac{d}{dx}\left(\int_{g(x)}^{f(x)} h(x,y)\,dy\right) = f'(x)h(x,f(x)) - g'(x)h(x,g(x)) + \int_{g(x)}^{f(x)} h_x(x,y)\,dy$$

kinematic and mass continuity equations 2

- literature is full of incomplete calculations of these equivalences
- . . . usually mixed in with small-parameter arguments about shallowness
- most ice sheet models use the mass continuity equation
- . . . but they could instead use the surface kinematical equation

standard recipe for ice sheet models

- the ingredients of a typical ice sheet model:
  1. numerical implementation of a stress balance: compute velocity (u, v, w)
  2. from the horizontal velocity (u, v) and the surface balance, do time-step of mass continuity equation to get H_t
  3. update surface elevation (and bed elevation)
  4. decide on time-step, and repeat at 1.

Outline

introduction: view from outside glaciology

shallow ice sheets

mass continuity

shelves and streams

flow model II: shallow shelf approximation (SSA) stress balance

SSA model applies very well to ice shelves

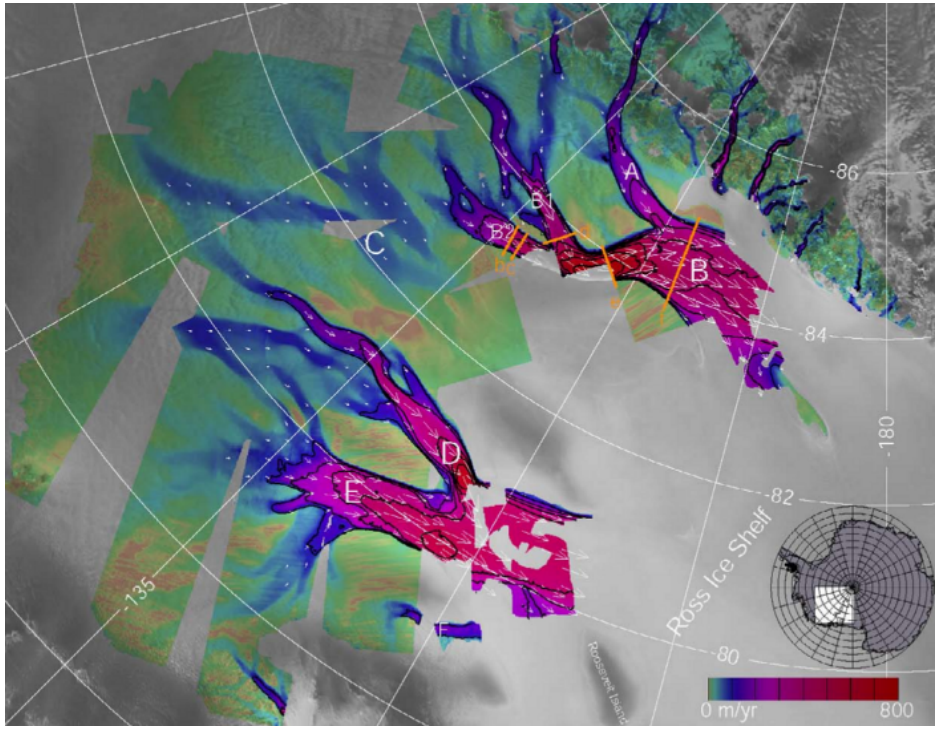
- . . . for parts away from grounding lines
- . . . and away from calving fronts

Ekström ice shelf(Hans Grobe)

shallow shelf approximation stress balance 2

SSA also applies reasonably well to ice streams

- . . . with modest bed topography
- . . . and weak bed strength[2]
- imperfect near shear margins and grounding lines

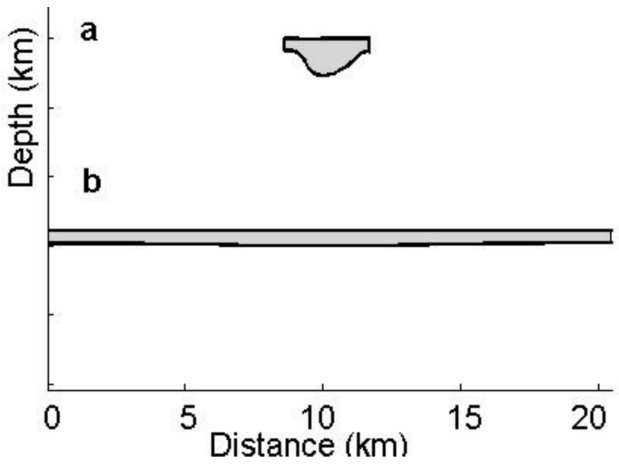


surface velocity for Siple Coast ice streams, Antarctica

[2] energy conservation (esp. ice temperature and basal melt) and subglacial hydrology (esp. subglacial water pressure) are major aspects of ice stream flow . . . but not addressed here

what is, *and is not*, an ice stream?

- ice streams
  - slide (100 to 1000 m a^{-1})
  - have concentrated vertical shear in thin layer near base
- "outlet glaciers"
  - fast surface speed (up to 10 km a^{-1})
  - uncertain how much is sliding
  - substantial vertical shear "up" in the ice column,
  - not-at-all flat bed topography
  - soft, temperate ice may play a big role
- few simplifying assumptions are appropriate for outlet glaciers



Jakobshavns Isbrae (**a**) and Whillans Ice Stream (**b**); plotted without vertical exaggeration (Truffer and Echelmeyer (2003), *Of isbrae and ice streams*)

SSA stress balance equation

- only plane flow case ("flow line") here
- the stress balance equation which determines velocity in an *ice stream*:

$$\boxed{\left(2A^{-1/n}H|u_x|^{1/n-1}u_x\right)_x - C|u|^{m-1}u = \rho g H h_x} \qquad (2)$$

- the red term inside parentheses is the vertically-integrated "longitudinal" or "membrane" stress
- the blue term is basal resistance
- the green term is driving stress
- derived originally by Morland (1987), MacAyeal (1989)
- *how to think about this equation*?
- *how do you solve it numerically*?

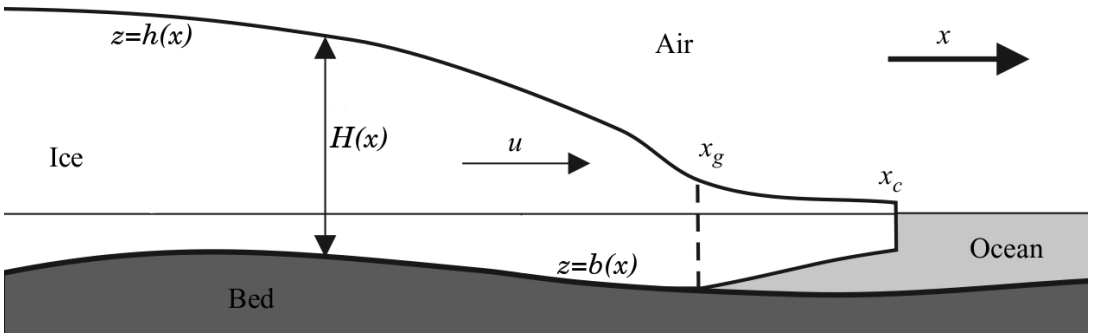
flow line model: from stream to shelf

$$u = u_0 \qquad \text{at } x = 0$$

$$\left.\begin{aligned} \left(2A^{-1/n}H|u_x|^{1/n-1}u_x\right)_x - C|u|^{m-1}u &= \rho g H h_x \\ h &= H + b \end{aligned}\right\} \qquad \text{on } 0 < x < x_g$$

$$\left.\begin{aligned} \left(2A^{-1/n}H|u_x|^{1/n-1}u_x\right)_x + 0 &= \rho g H h_x \\ h &= (1 - \rho/\rho_w)H \end{aligned}\right\} \qquad \text{on } x_g < x < x_c$$

$$2A^{-1/n}H|u_x|^{1/n-1}u_x = \frac{1}{2}\rho(1 - \rho/\rho_w)gH^2 \qquad \text{at } x = x_c$$



flotation criterion and grounding line

- the inequality "$\rho H < -\rho_w b$" is the flotation criterion
- at the grounding line $x = x_g$ the above inequality switches
- ...and the driving stress switches form:
  - on the grounded side we know $\rho H > -\rho_w b$ so

$$\rho g H h_x = \rho g H (H_x + b_x)$$

  - on the floating side we know $\rho H < -\rho_w b$ so $h = (1 - \rho/\rho_w) H$ and so

$$\rho g H h_x = \rho (1 - \rho/\rho_w) g H H_x$$

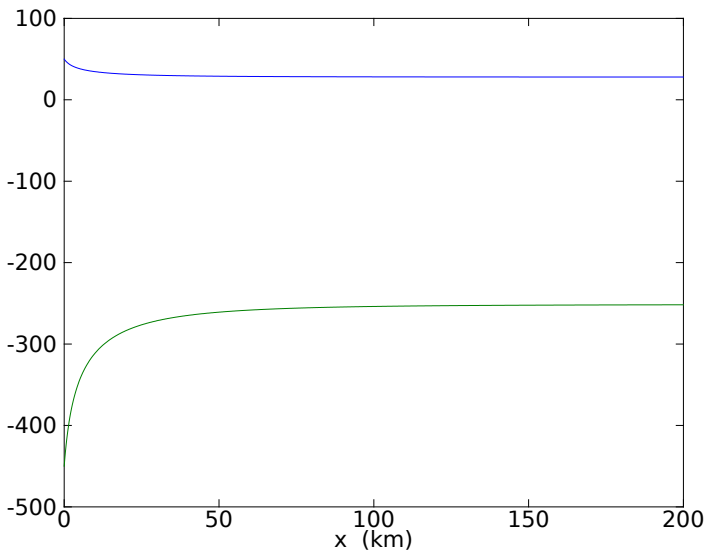
- also: H, u, u_x are all continuous at $x = x_g$

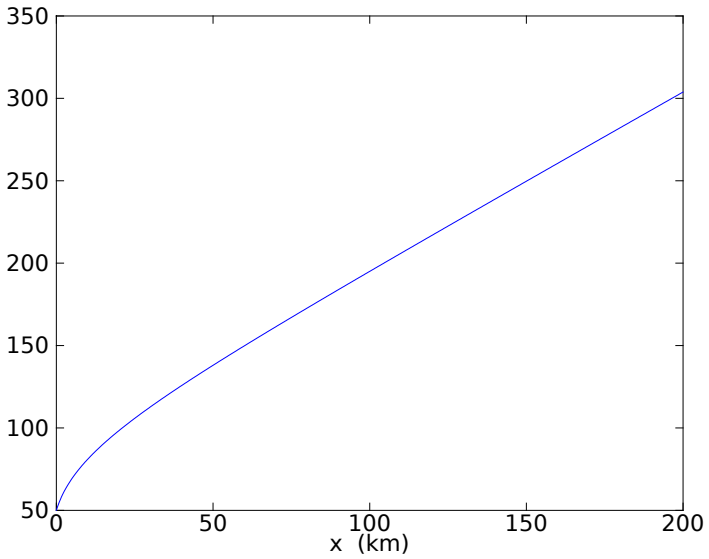
exact velocity and thickness for steady ice shelf

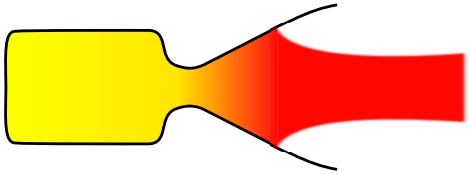
- limited goal here: describe a steady state, 1D ice shelf
- there is a nice by-hand result (next slide): the thickness and velocity in the ice shelf can be completely determined in terms of the
  1. ice thickness H_g at the grounding line and
  2. ice velocity u_g at the grounding line
- we will use this to
  - understand the SSA better
  - verify a numerical SSA code

exact velocity and thickness for steady ice shelf 2

see `testshelf.m`







numerically solving the SSA stress balance

- here we fix ice thickness $H(x)$ and find the velocity numerically
- the stress balance is a nonlinear equation in the velocity:

$$\left(2A^{-1/n}H|u_x|^{1/n-1}u_x\right)_x - C|u|^{m-1}u = \rho g H h_x$$

- iteration is needed
- I'll describe the numerical method for a shelf *or* stream, but only give a code for an ice shelf

numerically solving the SSA stress balance 2

- coefficient $\bar{\nu} = A^{-1/n}|u_x|^{1/n-1}$ is the "effective viscosity":

$$(2\,\bar{\nu}\,Hu_x)_x - C|u|^{m-1}u = \rho g H h_x$$

- *simplest iteration idea*: use old effective viscosity to get new velocity solution, and repeat until things stop changing
  - this is "Picard" iteration
  - Newton iteration is a superior alternative
- specifically:
  - last iterate $u^{(k-1)}$
  - define $W^{(k-1)} = 2\bar{\nu}H = 2A^{-1/n}|u_x^{(k-1)}|^{1/n-1}H$
  - current iterate (unknown) $u^{(k)}$
  - solve repeatedly:

$$\left(W^{(k-1)}u_x^{(k)}\right)_x - C|u^{(k-1)}|^{m-1}u^{(k)} = \rho g H h_x$$

solving the "inner" linear problem

- abstract the problem:

$$(W(x)\,u_x)_x - \alpha(x)\,u = \beta(x)$$

on $0 < x < L$, with boundary conditions

$$u(0) = V, \qquad u_x(L) = \gamma$$

- an *elliptic* PDE boundary value problem
- $W(x)$, $\alpha(x)$, $\beta(x)$ are known functions in the SSA context:
  - both $W(x)$ and $\alpha(x)$ come from previous iteration
  - $\beta(x)$ is driving stress

where do you get an initial guess $u^{(0)}$?

- *for floating ice*, a possible initial guess for velocity comes from assuming a uniform strain rate:

$$u^{(0)}(x) = \gamma(x - x_g) + u_g$$

  where γ is the value of u_x found from calving front stress imbalance
- *for grounded ice*, a possible initial guess for velocity is to assume ice is held by basal resistance only:

$$u^{(0)}(x) = \left(-C^{-1}\rho g H h_x\right)^{1/m}$$

numerics of the "inner" linear problem

- suppose $j = 1, 2, \ldots, J+1$, where $x_1 = x_g$ and $x_{J+1} = x_c$ are endpoints
- $W(x)$ is needed on the staggered grid; the approximation is:

$$\frac{W_{j+1/2}(u_{j+1} - u_j) - W_{j-1/2}(u_j - u_{j-1})}{\Delta x^2} - \alpha_j u_j \stackrel{*}{=} \beta_j$$

- left-hand boundary condition: $u_1 = V$ given
- right-hand boundary condition ("$u_x(L) = \gamma$"):
  - introduce notional point x_{J+2}
  - $$\frac{u_{J+2} - u_J}{2\Delta x} = \gamma$$
  - using equation $*$ in $j = J+1$ case, eliminate u_{J+2} variable "by-hand" before coding numerics

numerics of the "inner" linear problem 2

- so SSA stress balance has form $A\mathbf{x} = \mathbf{b}$, namely:

$$\begin{bmatrix} 1 & & & & & \\ W_{3/2} & A_{22} & W_{5/2} & & & \\ & W_{5/2} & A_{33} & & & \\ & & \ddots & \ddots & & \\ & & W_{J-1/2} & A_{JJ} & W_{J+1/2} \\ & & & A_{J+1,J} & A_{J+1,J+1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_J \\ u_{J+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \beta_2 \Delta x^2 \\ \beta_3 \Delta x^2 \\ \vdots \\ \beta_J \Delta x^2 \\ b_{J+1} \end{bmatrix}$$

- with diagonal entries

$$A_{22} = -(W_{3/2} + W_{5/2} + \alpha_1 \Delta x^2)$$

$$A_{33} = -(W_{5/2} + W_{7/2} + \alpha_2 \Delta x^2)$$

  and so on, up to A_{JJ},
- with special cases in last equation:

$$A_{J+1,J} = 2W_{J+1/2}$$

$$A_{J+1,J+1} = -(2W_{J+1/2} + \alpha_{J+1} \Delta x^2)$$

$$b_{J+1} = -2\gamma \Delta x W_{J+3/2} + \beta_{J+1} \Delta x^2$$

- this is a *tridiagonal* system

numerics of the “inner” linear problem 3

```
function u = flowline(L,J,gamma,W,alpha,beta,V0)

dx = L / J;
rhs = dx^2 * beta(:);
rhs(1) = V0;
rhs(J+1) = rhs(J+1) - 2 * gamma * dx * W(J+1);

A = sparse(J+1,J+1);
A(1,1) = 1.0;
for j=2:J
  A(j,j-1:j+1) = [ W(j-1), -(W(j-1) + W(j) + alpha(j) * dx^2), W(j) ];
end
A(J+1,J) = W(J) + W(J+1);
A(J+1,J+1) = - (W(J) + W(J+1) + alpha(J+1) * dx^2);

scale = full(max(abs(A),[],2));
for j=1:J+1,  A(j,:) = A(j,:) ./ scale(j);  end
rhs = rhs ./ scale;

u = A \ rhs;
```

flowline.m

testing the “inner” linear code

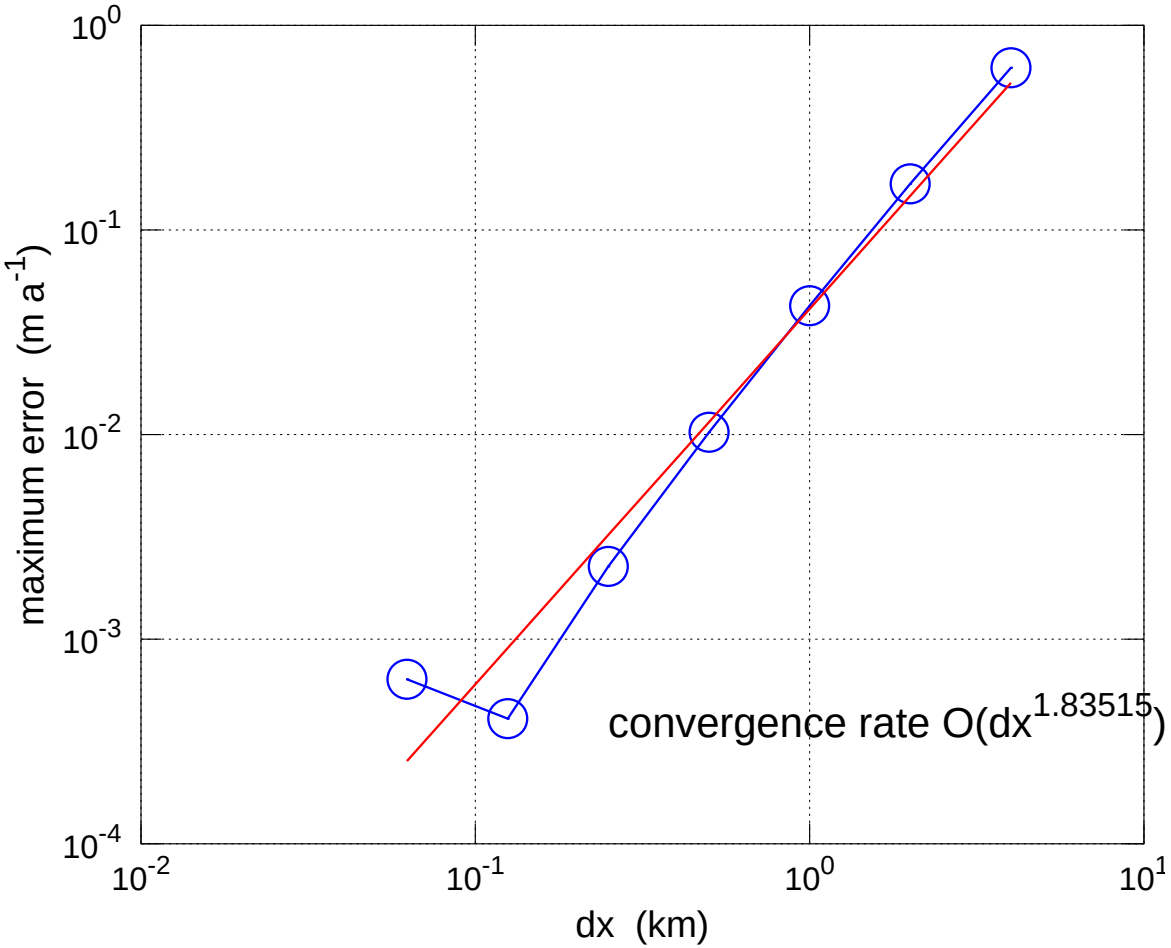
- before proceeding to solve nonlinear SSA problem, we can test the “abstracted” code `flowline.m`
- test by “manufacturing” solutions
  - see `testflowline.m`; not shown
- results:
  - converges at optimal rate $O(\Delta x^2)$

numerical: SSA

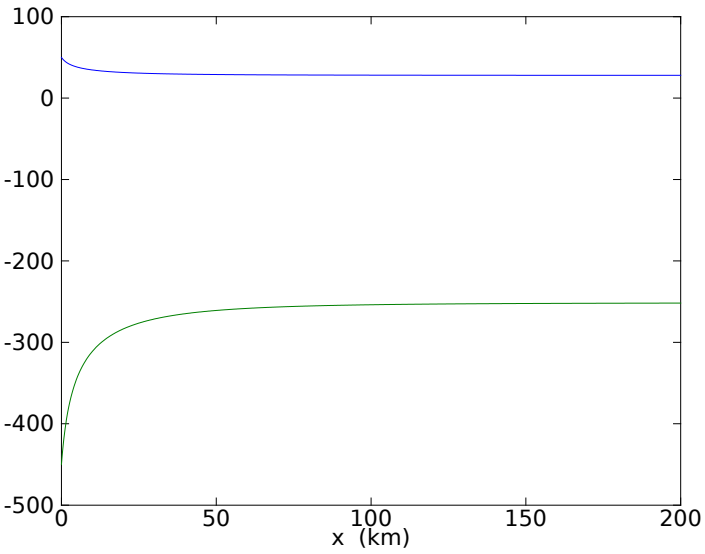
```
function [u,u0] = ssaflowline(p,J,H,b,ug,initchoice)

if nargin ~= 6, error('exactly 6 input arguments required'), end

dx = p.L / J;
x = (0:dx:p.L)';
xstag = (dx/2:dx:p.L+dx/2)';

alpha = p.C * ones(size(x));
h = H + b;
hx = regslope(dx,h);
beta = p.rho * p.g * H .* hx;
gamma = ( 0.25 * p.A^(1/p.n) * (1 - p.rho/p.rhow) *...
          p.rho * p.g * H(end) )^p.n;

u0 = ssainit(p,x,beta,gamma,initchoice);
u = u0;

Hstag = stagav(H);
tol = 1.0e-14;
eps_reg = (1.0 / p.secpera) / p.L;
maxdiff = Inf;
W = zeros(J+1,1);
iter = 0;
while maxdiff > tol
  uxstag = stagslope(dx,u);
  sqr_ux_reg = uxstag.^2 + eps_reg^2;
  W(1:J) = 2 * p.A^(-1/p.n) * Hstag .* sqr_ux_reg.^(((1/p.n)-1)/2.0);
  W(J+1) = W(J);

  unew = flowline(p.L,J,gamma,W,alpha,beta,ug);
  maxdiff = max(abs(unew-u));
  u = unew;
  iter = iter + 1;
end

  function fav = stagav(f)
  fav = 0.5 * (f(1:end-1) + f(2:end));

  function slope = regslope(dx,f)
  J = length(f) - 1;
  slope = [(f(2)-f(1))/dx; (f(3:J+1)-f(1:J-1))/(2*dx); (f(J+1)-f(J))/dx];
```

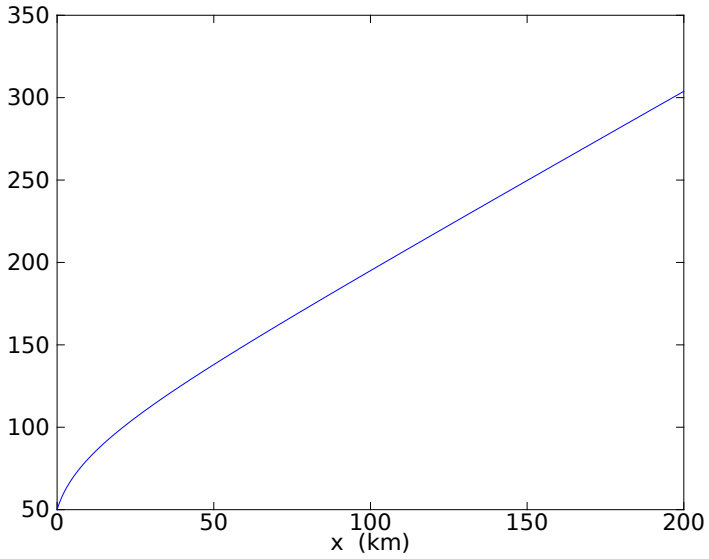
numerical thickness and velocity for steady ice shelf

lines below are a convergence analysis of `testshelf.m`, which calls `ssaflowline.m`:



SSA model output

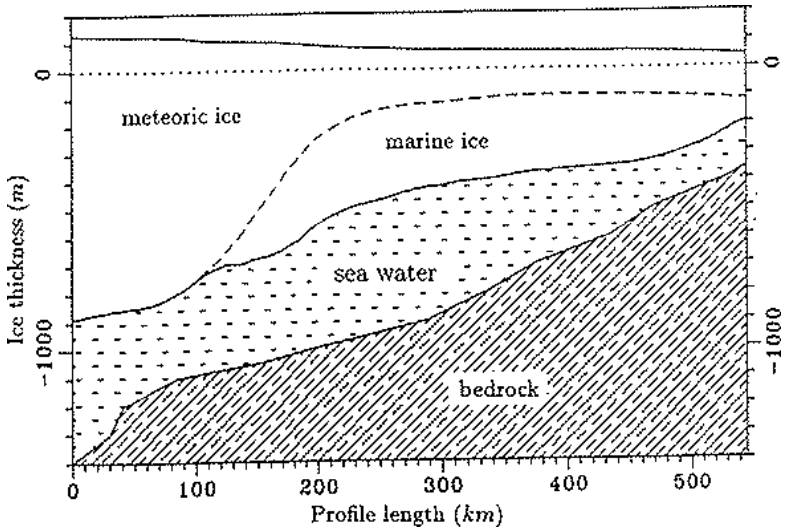




- *this looks suspiciously like figures for the exact solution . . .*
- yes

realistic ice shelf modeling

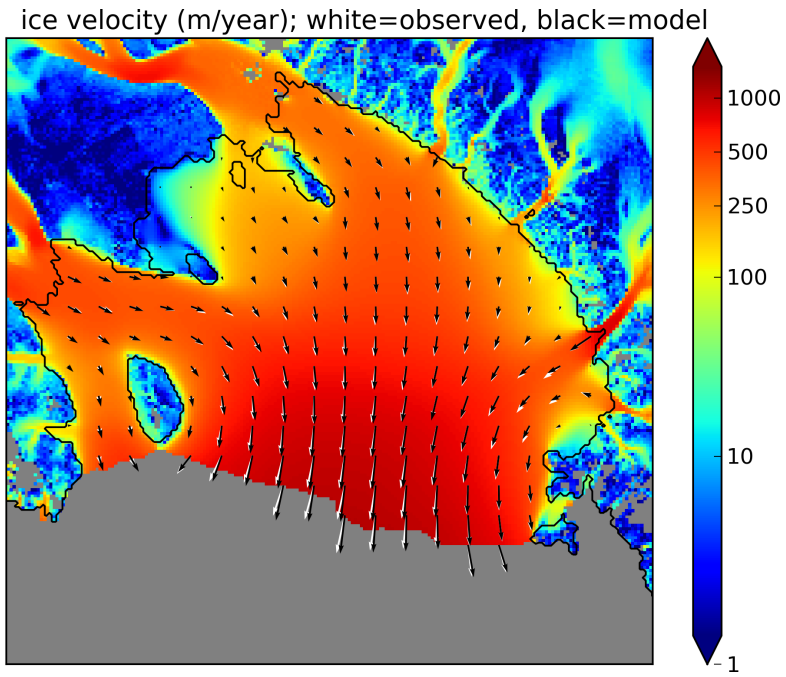
- flow lines are never very realistic
- you can add parameterized “side drag” . . .
- also, ice shelves have surprises:
  - high basal melt near grounding lines
  - marine ice can freeze-on at bottom (below)
  - “reverse slope” bed instability and WAIS . . .

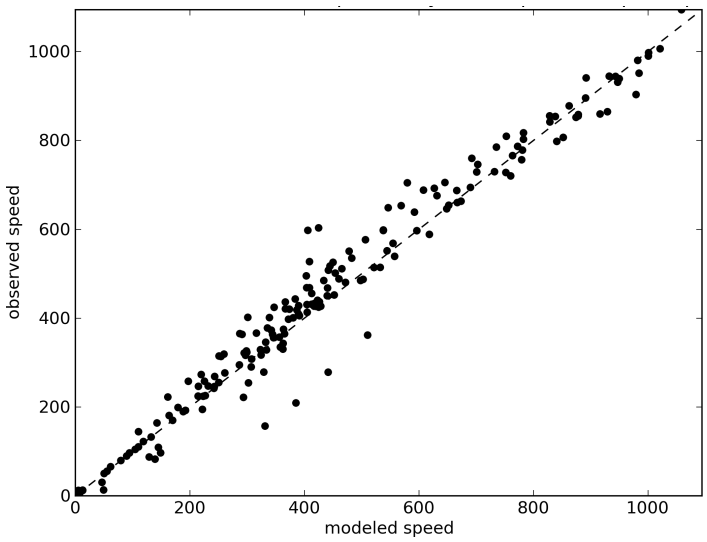


from Grosfeld & Thyssen 1994

ice shelf modeling in 2D

- nonetheless “diagnostic” (static geometry) ice shelf modeling in 2d has been quite successful
- observed surface velocities validate SSA stress balance model
  - e.g. Ross ice shelf example below using PISM
  - . . . but many models can do this





numerical solution of stress balances: a summary

- stress balance equations (e.g. SSA or Stokes) determine velocity from geometry and boundary conditions
  - nonlinear so iteration is necessary
  - at each iteration a sparse matrix “inner” problem is solved
    . . . give it to a matrix solver software package

- general principles:
  - modularize your code
  - test the parts

the mass continuity equation: a summary

- the *mass continuity equation* is

$$H_t = M - \nabla \cdot (\mathbf{u} H)$$

- the numerical nature of this equation depends on the stress balance:
  - the equation is a diffusion for frozen bed, large scale flows (i.e. SIA)
  - it is *not* very diffusive for membrane stresses and no basal resistance (e.g. SSA for ice shelves)
  - it is diffusive for ice streams (but how much?)
  - there is *not* much helpful theory on this transport problem
  - . . . maybe you will help find this theory!