Dynamics and thermodynamics of Glaciers

Solutions to Exercise

1 Flow speeds

You can either use the fact that the flow velocity on the center line of a cylindrical channel is eight times slower than in an ice sheet of the same ice thickness (Eq. 32 in the script) or you estimate the radius by eye-balling the map. One method underestimates the flow speed, the other two overestimate it. This little python snippet does the job:

```
#!/usr/bin/env python

import numpy as np

rho = 910
g = 9.81
alpha = 1.7
A = 2.4e-24
n = 3
secpera = 31556926

R = 1500
H = 900

f = np.sin(alpha*np.pi/180.)

u_def_channel = 2 * A * (0.5 * rho * g * f)**n * ((R**(n+1))/(n+1)) * secpera
u_def_slab = 2 * A * (rho * g * f)**n * ((H**(n+1))/(n+1)) * secpera

print('channel speed for radius {}m: {:3.0f} m/yr'.format(R, u_def_channel))
print('channel speed for thickness {}m: {:3.0f} m/yr'.format(H, u_def_channel/8))
print('slab speed for thickness {}m: {:3.0f} m/yr'.format(H, u_def_slab))
```

2 Mass flux

Yes Carl is right because in the shearing case the surface velocity v_h is given by

$$v_h = \int_0^h v \, \mathrm{d}z = \frac{A_0}{n+1} H^{n+1}$$

and the vertically-average velocity by

$$\bar{v} = \frac{1}{H}\int_0^v \mathrm{d}z = \frac{1}{H}\int_0^h A_0 \frac{1}{n+1} h^{n+1}\,\mathrm{d}z = \frac{A_0}{H}\frac{1}{n+2}H^{n+2} = \frac{A_0}{n+2}H^{n+1}$$

It thus follows that the ratio $\bar{v}/v_h$ is

$$\frac{\bar{v}}{v_h} = \frac{n+1}{n+2} = 0.8$$

for $n = 3$.

3 Surface evolution

1. In simple terms the surface responds to changes in dynamics (flux divergence) and climate (climatic mass balance)
2.
   - in the accumulation zone, the surface rises 1 m from September to March, and lowers by 1.5 m from March to September. At the end of the hydrological year, the surface is thus 0.5 m lower.
   - in the ablation zone, the surface rises 1 m from September to March, and lowers by 1 m from March to September. At the end of the hydrological year, the surface is thus the same.
3. because the glacier is shielded from ablation and accumulation (i.e. $a_\mathrm{s} = 0$), the glaciers slowly things and spreads under its own weight.

4 Mass balance, surface evolution, and vertical velocity

Well, the example was taken slightly out of context, as the paper deals with rock glaciers. In this special case, the vertical velocity is indeed equal to the change of elevation with time. The term “spatial mass balance” is, however, an unlucky choice. Laser altimetry measures the surface elevation.

5 Climate history

The initial amplitude is $\Delta T_0 = 2^\circ$ C during an oscillation period of 50 years, thus $\omega = \frac{2\pi}{40\,\mathrm{a}} = 3.9821 \times 10^{-9}\,\mathrm{s}^{-1}$. With Equation (11) we get:

$$\frac{\Delta T(h)}{\Delta T_0} = \exp\left(-h\sqrt{\frac{\omega}{2\kappa}}\right).$$

The thermal diffusivity can be obtained from Equations (2) and (3)

$$\kappa(-3)^\circ\,\mathrm{C} = \frac{k(-3^\circ\,\mathrm{C})}{\rho\, c(-3^\circ\,\mathrm{C})} = \frac{2.1\,\mathrm{W\,m^{-1}\,K^{-1}}}{910\,\mathrm{kg\,m^{-3}} \cdot 2105.7\,\mathrm{J\,kg^{-1}\,K^{-1}}} = 1.34 \times 10^{-6}\,\mathrm{m^2\,s^{-1}},$$

or, just the value given in the script, $1.09 \cdot 10^{-6}\,\mathrm{m^2\,s^{-1}}$ (the result then may slightly vary). Therefore we get

$$h = -\sqrt{\frac{2\kappa}{\omega}}\ln\left(\frac{\Delta T(h)}{\Delta T_0}\right) = -\sqrt{\frac{2 \cdot 1.34 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}}{3.9821 \times 10^{-9}\,\mathrm{s^{-1}}}}\ln\left(\frac{0.01}{2}\right) \approx 137\,\mathrm{m}.$$

6 Cold content

The cold content can be obtained by

$$E = -\int_0^z \rho c_p T(z)\,\mathrm{d}z'$$

This can be approximated by the area under the depth profile:

$$E \approx \frac{1}{2}\rho c_p \Delta T \Delta z = 105 \cdot 10^5\,\mathrm{J\,kg^{-1}},$$

where $\rho = 500\,\mathrm{kg\,m^{-3}}$ and $c_p = 2009\,\mathrm{J\,kg^{-1}\,K^{-1}}$. Divide by the latent heat of fusion $L = 3.34 \cdot 10^5\,\mathrm{J\,kg^{-1}}$ and you get $32\,\mathrm{kg\,m^{-2}} = 32$ mm w.e..

7 Melting temperature depression

From Figure (9) we see that the pressure melting temperature is $T_\mathrm{m} = -0.34°$ C. We have $p = 910\,\mathrm{kg\,m^3} \cdot 9.81\,\mathrm{m\,s^{-2}} \cdot 300\,\mathrm{m} + 75'000\,\mathrm{Pa} = 2753130\,\mathrm{Pa}$. Using Equation (14) we obtain

$$\gamma = -\frac{T_\mathrm{m} - T_\mathrm{tp}}{p - p_\mathrm{tp}} = \frac{273.15\,\mathrm{K} - 0.34\,\mathrm{K} - 273.16\,\mathrm{K}}{2753130\,\mathrm{Pa} - 611.73\,\mathrm{Pa}} = 1.202 \times 10^{-7}\,\mathrm{K\,Pa^{-1}}.$$

In the literature we find the values $\gamma = 7.42 \times 10^{-8}\,\mathrm{K\,Pa^{-1}}$ for pure ice and air-free water and $\gamma = 9.8 \times 10^{-7}\,\mathrm{K\,Pa^{-1}}$ for pure ice and air-saturated water. The value calculated for Gornergletscher is even higher than γ for air-saturated water which means we have ice with air-saturated water.

8 Lake Vostok

1. Advection and diffusion
2. The Péclet number Pe is a measure of the relative importance of advection and diffusion.
3. First we calculate the pressure melting point at the base

$$\begin{aligned} T_m &= T_{tp} - \gamma\,(p - p_\mathrm{tp}) \\ &= 273.16\,\mathrm{K} - 7.42 \times 10^{-8}\,\mathrm{K\,Pa^{-1}}\left(910\,\mathrm{kg\,m^3} \cdot 9.81\,\mathrm{m\,s^{-2}} \cdot 3300\,\mathrm{m} - 611.73\,\mathrm{Pa}\right) \\ &\approx 271\,\mathrm{K} \end{aligned}$$

Then use Equation (21) from the script:

$$T(z) = T_s + \frac{\sqrt{\pi}}{2} l \left(\frac{dT}{dz}\right)_B \left[\mathrm{erf}\left(\frac{z}{l}\right) - \mathrm{erf}\left(\frac{H}{l}\right)\right].$$

In order for a lake to form we need $T(0\,\mathrm{m}) = T_\mathrm{m}$.

$$\begin{aligned} T(0\,\mathrm{m}) &= T_s + \frac{\sqrt{\pi}}{2} l \left(\frac{dT}{dz}\right)_B \left[\mathrm{erf}\left(\frac{z}{l}\right) - \mathrm{erf}\left(\frac{H}{l}\right)\right] \\ &= T_s + \frac{\sqrt{\pi}}{2} l \left(\frac{G}{k}\right) \left[\mathrm{erf}\left(\frac{z}{l}\right) - \mathrm{erf}\left(\frac{H}{l}\right)\right] \end{aligned}$$

The geothermal flux at the base thus must be equal or larger:

$$G \geq \frac{T_s - T(0\,\mathrm{m})}{\frac{\sqrt{\pi}}{2} l \left(\frac{1}{k}\right)\left[-\mathrm{erf}\left(\frac{H}{l}\right)\right]} \approx 0.05\,\mathrm{W\,m^2}$$